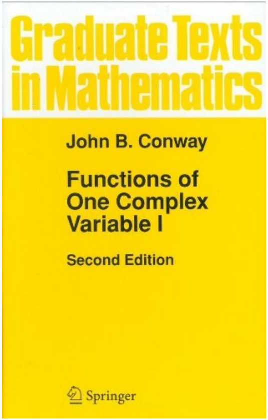
Complex Analysis

Chapter III. Elementary Properties and Examples of Analytic Functions

III.2. Analytic Functions—Proofs of Theorems



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Proposition III.2.2. If $f : G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Proof. We have

$$\begin{aligned} \lim_{z \to a} |f(z) - f(a)| &= \lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|} |z-a| \\ &= \lim_{z \to a} \left| \frac{f(z) - f(a)}{z-a} \right| \lim_{z \to a} |z-a| \text{ since the limit} \\ &\quad \text{of a product is the product of the limits} \\ &\quad \text{(provided the component limits exist)} \\ &= |f'(a)| \cdot 0 = 0. \end{aligned}$$

Therefore $\lim_{z \to a} f(z) = f(a)$ and f is continuous at a, as claimed. □

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Chain Rule

Chain Rule. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$ for all $z \in G$.

Proof. Fix $z_0 \in G$ and choose $r > 0$ such that $B(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\} \subset G$. Since $g \circ f$ is continuous at z_0 by Proposition III.2.2 (and properties of continuous functions), it is sufficient to show that if $0 < |h_n| < r$ and $\lim h_n = 0$, then $\lim_{n\to\infty}((g(f(z_0 + h_n)) - g(f(z_0))/h_n)$ exists and equals $g'(f(z_0))f'(z_0)$.

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Case 1. Suppose $f(z_0) \neq f(z_0 + h_n)$ for all n. Then

$$\frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n} = \frac{g(f(z_0 + h_n)) - g(f(z_0))}{f(z_0 + h_n) - f(z_0)} \frac{f(z_0 + h_n) - f(z_0)}{h_n}.$$

By Proposition III.2.2, $\lim_{n\to\infty}(f(z_0 + h_n) - f(z_0)) = 0$, so the limit is $g'(f(z_0))f'(z_0)$.

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Chain Rule (continued)

Chain Rule. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$ for all $z \in G$.

Proof (continued). Case 2. Suppose $f(z_0) = f(z_0 + h_n)$ for infinitely many n. Then write $\{h_n\}$ as the union of two sequences $\{k_n\}$ and $\{\ell_n\}$ where $f(z_0) \neq f(z_0 + k_n)$ and $f(z_0) = f(z_0 + \ell_n)$ for all n. Since f is differentiable, $f'(z_0) = \lim_{n\to\infty} \frac{f(z_0+\ell_n)-f(z_0)}{\ell_n} = \lim_{n\to\infty} \frac{0}{\ell_n} = 0$. Also, $\lim_{n\to\infty} \frac{g\circ f(z_0+\ell_n)-g\circ f(z_0)}{\ell_n} = \lim_{n\to\infty} \frac{0}{\ell_n} = 0$. By Case 1, $\lim_{n\to\infty} \frac{g\circ f(z_0+k_n)-g\circ f(z_0)}{k_n} = g'(f(z_0))f'(z_0)$. Since $f(z_0) = f(z_0 + h_n)$ for infinitely many h_n and f is continuous at z_0, then $f'(z_0) = 0$. Therefore $\lim_{n\to\infty} \frac{g\circ f(z_0+k_n)-g\circ f(z_0)}{h_n} = 0 = g'(f(z_0))f'(z_0)$. Combining Case 1 and case 2, the result follows. □

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Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(a) for $k \geq 1$ the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$$

has radius of convergence R,

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k \geq 1$ and $|z-a| < R$, and

(c) for $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Without loss of generality, assume $a = 0$. **(a)** We prove (a) for $k = 1$ and the result follows in general by induction. By definition, $1/R = \limsup |a_n|^{1/n}$.

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Proof (continued). With $k = 1$, we consider $\sum_{n=1}^{\infty} na_n z^{n-1}$ and need to show that $1/R = \limsup |na_n|^{1/(n-1)}$. By L'Hopital's Rule, $\lim_{n\to\infty} n^{1/(n-1)} = 1$. So, by Exercise III.2.2,

$$\limsup |na_n|^{1/(n-1)} = \lim n^{1/(n-1)} \limsup |a_n|^{1/(n-1)} = \limsup |a_n|^{1/(n-1)}.$$

We now need to show that $1/R = \limsup |a_n|^{1/(n-1)}$. Let $1/R' = \limsup |a_n|^{1/(n-1)}$. Then R' is the radius of convergence of

$$\sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n.$$

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Proposition III.2.5 (continued 2)

Proposition III.2.5. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

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Proof (continued). Now, $z\sum_{n=0}^{\infty} a_{n+1}z^n + a_0 = \sum_{n=0}^{\infty} a_n z^n$ and so if $|z| < R'$ then $\sum_{n=0}^{\infty} |a_n z^n| = |a_0| + |z|\sum_{n=0}^{\infty} |a_{n+1}z^n| < \infty$. So $R' \leq R$. If $|z| < R$ and $z \neq 0$ then $\sum_{n=0}^{\infty} |a_n z^n| < \infty$ and

$$\sum_{n=0}^{\infty} |a_{n+1}z^n| = \frac{1}{|z|}\sum_{n=0}^{\infty} |a_n z^n| - \frac{1}{|z|}|a_0| < \infty$$

and so $R \leq R'$. Therefore $R = R'$. $\square$

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Proposition III.2.5 (continued 3)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(b) The function f is infinitely differentiable on $B(a;R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k \geq 1$ and $|z-a| < R$.

Proof. (b) For $|z| < R$ put $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$, $s_n(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$ (so $f(z) = s_n(z) + R_n(z)$). Fix $w \in B(0;R) = \{z \mid |z-0| < R\}$ and fix r with $|w| < r < R$. We will show $f'(w) = g(w)$.

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$$\frac{f(z) - f(w)}{z - w} - g(w) = \left(\frac{s_n(z) - s_n(w)}{z - w} - s_n'(w)\right) + (s_n'(w) - g(w)) + \left(\frac{R_n(z) - R_n(w)}{z - w}\right). \quad (2.8)$$

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$$\frac{f(z)-f(w)}{z-w} - g(w) = \left(\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right) + \left(s_n'(w) - g(w)\right) + \left(\frac{R_n(z)-R_n(w)}{z-w}\right). \quad (2.8)$$

Proposition III.2.5 (continued 4)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

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Proof (continued). (b) Now

$$\frac{R_n(z) - R_n(w)}{z-w} = \frac{1}{z-w}\left(\sum_{k=n+1}^{\infty} a_k(z^k - w^k)\right) = \sum_{k=n+1}^{\infty} a_k\left(\frac{z^k - w^k}{z-w}\right).$$

But $\dfrac{|z^k - w^k|}{|z-w|} = |z^{k-1} + z^{k-2}w + \cdots + zw^{k-2} + w^{k-1}| \leq kr^{k-1}$ (since $w, z < r$). Hence,

$$\left|\frac{R_n(z) - R_n(w)}{z-w}\right| \leq \sum_{k=n+1}^{\infty} |a_k| kr^{k-1}.$$

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Proposition III.2.5 (continued 5)

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Proof (continued). (b) Since $r < R$, then $\sum_{n=1}^{\infty} |a_k| k r^{k-1}$ converges (consider part (a) with $|z| = r < R$). So for any $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\left| \frac{R_n(z) - R_n(w)}{z-w} \right| < \frac{\varepsilon}{3}$ (here, $z \in B(w; \delta_1)$). By the definitions of s_n and g, $\lim_{n\to\infty} s_n'(w) = g(w)$, so there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|s_n'(w) - g(w)| < \varepsilon/3$. Let $n = \max\{N_1, N_2\}$. Then there is $\delta_2 > 0$ such that $\left| \frac{s_n(z) - s_n(w)}{z-w} - s_n'(w) \right| < \frac{\varepsilon}{3}$ whenever $0 < |z-w| < \delta_2$. With $z \in B(w; \delta)$, $z \neq w$, where $\delta = \min\{\delta_1, \delta_2\}$ we have from (2.8) that $\left| \frac{f(z) - f(w)}{z-w} \right| < \varepsilon$. That is, $f'(w) = g(w)$. □

Proposition III.2.5 (continued 5)

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Proof (continued). (b) Since $r < R$, then $\sum_{n=1}^{\infty} |a_k| k r^{k-1}$ converges (consider part (a) with $|z| = r < R$). So for any $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\left|\frac{R_n(z)-R_n(w)}{z-w}\right| < \frac{\varepsilon}{3}$ (here, $z \in B(w; \delta_1)$). By the definitions of s_n and g, $\lim_{n\to\infty} s_n'(w) = g(w)$, so there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|s_n'(w) - g(w)| < \varepsilon/3$. Let $n = \max\{N_1, N_2\}$. Then there is $\delta_2 > 0$ such that $\left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| < \frac{\varepsilon}{3}$ whenever $0 < |z-w| < \delta_2$. With $z \in B(w; \delta)$, $z \neq w$, where $\delta = \min\{\delta_1, \delta_2\}$ we have from (2.8) that $\left|\frac{f(z)-f(w)}{z-w}\right| < \varepsilon$. That is, $f'(w) = g(w)$. $\square$

Proposition III.2.5 (continued 6)

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. From part (a), we have $f^{(k)}(0) = k!a_k$ and the result follows. □

Proposition III.2.10.

Proposition III.2.10. If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $a \in \mathbb{C}$, then f is constant.

Proof. Fix $z_0 \in G$ and denote $\omega_0 = f(z_0)$. Let $A = \{z \in G \mid f(z) = \omega_0\}$. Let $z \in G$ and $\{z_n\} \subset A$ where $z = \lim z_n$. Since $f(z_n) = \omega_0$ for $n \in \mathbb{N}$ (each $z_n \in A$) and f is continuous (since f is differentiable; Proposition III.2.2) then $f(z) = f(\lim z_n) = \lim f(z_n) = \omega_0$, and so $z \in A$. So A contains all of its limit points and by Proposition II.3.4 A is closed in G.

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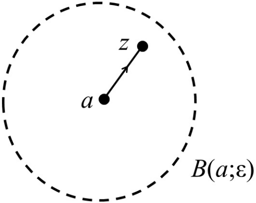
Proof. Fix $z_0 \in G$ and denote $\omega_0 = f(z_0)$. Let $A = \{z \in G \mid f(z) = \omega_0\}$. Let $z \in G$ and $\{z_n\} \subset A$ where $z = \lim z_n$. Since $f(z_n) = \omega_0$ for $n \in \mathbb{N}$ (each $z_n \in A$) and f is continuous (since f is differentiable; Proposition III.2.2) then $f(z) = f(\lim z_n) = \lim f(z_n) = \omega_0$, and so $z \in A$. So A contains all of its limit points and by Proposition II.3.4 A is closed in G. Next, fix $a \in A$ and let $\varepsilon > 0$ be such that $B(a;\varepsilon) \subset G$ (since G is open). If $x \in B(a;\varepsilon)$, set $g(t) = f(tz + (1-t)a)$ for $t \in [0,1]$:



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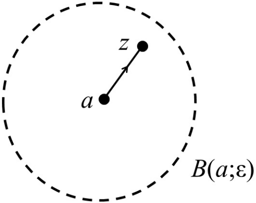

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Proof (continued). Now $g : [0,1] \to \mathbb{C}$ is a composition of differentiable $f : G \to \mathbb{C}$ with $h : [0,1] \to G$ where $h(t) = tz + (1-t)a$. To differentiate g, we need a version of the Chain Rule which is applicable to this setting—this is given in Appendix A on page 304 in Proposition A.4.
So

$$g'(t) = \lim_{t \to s} \frac{g(t) - g(s)}{t - s} = f'(tz + (1-t)a)(z-a) = 0$$

since $f'(z) = 0$ for $z \in G$. So $g'(t) = 0$ for $t \in [0,1]$ and hence (by Proposition A.3 of Appendix A on page 303, as applied to $g : [0,1] \to \mathbb{C}$) we have that g is constant. Therefore, $f(z) = g(1) = g(0) = f(a) = \omega_0$. Hence $B(a;\varepsilon) \subset A$ and A is open in G.
Since A is both open and closed in G, $a \neq \varnothing$ (since $a \in A$), and G is connected, then $A = G$. That is, f is constant on G. □

Proposition III.2.10 (continued).

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Hence $B(a; \varepsilon) \subset A$ and A is open in G.
Since A is both open and closed in G, $a \neq \varnothing$ (since $a \in A$), and G is connected, then $A = G$. That is, f is constant on G. □

Lemma III.2.A

Lemma III.2.A. Properties of e^z include:

(a) $e^{a+b} = e^a e^b$,

(b) $e^z \neq 0$ for all $z \in \mathbb{C}$,

(c) $\overline{e^z} = e^{\overline{z}}$, and

(d) $|e^z| = e^{\mathrm{Re}(z)}$.

Proof.
(a) Define $g(z) = e^z e^{a+b-z}$ for given $a, b \in \mathbb{C}$. Then $g'(z) = e^z e^{a+b-z} + e^z(-e^{a+b-z}) = 0$. So by Proposition 2.10, $g(z)$ is constant for all $z \in \mathbb{C}$. With $z = 0$, we have $g(0) = e^0 e^{a+b} = e^{a+b}$, so $e^z e^{a+b-z} = e^{a+b}$ for all $z \in \mathbb{C}$. With $z = b$ we have $e^b e^a = e^{a+b}$, as claimed.

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Lemma III.2.A (continued 1)

Lemma III.2.A. Properties of e^z include:

(c) $\overline{e^z} = e^{\overline{z}}$, and

(d) $|e^z| = e^{\text{Re}(z)}$.

Proof (continued).

(c) Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, then

$$\begin{aligned}
\overline{e^z} &= \overline{\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)} = \overline{\left(\lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n!}\right)} \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \overline{\left(\frac{z^n}{n!}\right)} \text{ since conjugation is continuous,} \\
&\quad \text{and Theorem I.2.A} \\
&= \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} = e^{\overline{z}}, \text{ as claimed.}
\end{aligned}$$

Lemma III.2.A (continued 2)

Lemma III.2.A. Properties of e^z include:

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(b) $e^z \neq 0$ for all $z \in \mathbb{C}$,

(c) $\overline{e^z} = e^{\overline{z}}$, and

(d) $|e^z| = e^{\mathrm{Re}(z)}$.

Proof (continued).
(d) By (c) we have

$$|e^z|^2 = e^z\overline{e^z} = e^z e^{\overline{z}} = e^{z+\overline{z}} \text{ by (a)}$$

and so $|e^z|^2 = e^{2\mathrm{Re}(z)}$ and $|e^z| = e^{\mathrm{Re}(z)}$, as claimed. □

Proposition III.2.19

Proposition III.2.19. If $G \subseteq \mathbb{C}$ is open and connected and f is a branch of $\log z$ on G, then the totality of branches of $\log z$ are the functions $\{f(z) + 2k\pi i \mid k \in \mathbb{Z}\}$.

Proof. First, if $g(z) = f(z) + 2k\pi i$, then

$$\exp(g(z)) = \exp(f(z) + 2k\pi i) = \exp(f(z))\exp(2k\pi i) = \exp(f(z)) = z,$$

so g is a branch of $\log z$.

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Proposition III.2.20

Proposition III.2.20. Let G and Ω be open subsets of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous where $f(G) \subseteq \Omega$ and $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) = 0$, then f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, then f is analytic.

Proof. Fix $a \in G$ and let $h \in \mathbb{C}$ such that $h \neq 0$ and $a + h \in G$. Then $a = g(f(a))$ (since $g(f(z)) = z$) and $a + h = g(f(a + h))$ implies $g(f(a)) \neq g(f(a + h))$ and so $f(a) \neq f(a + h)$ (or else these two *would* be equal since it would be g evaluated at the same point). So

$$1 = \frac{(a+h)-(a)}{h} = \frac{g(f(a+h)) - g(f(a))}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \frac{f(a+h) - f(a)}{h}.$$

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Proposition III.2.20 (continued)

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Proof (continued). The limit is of course 1, and since $\lim_{h \to 0}(f(a+h) - f(a)) = 0$, then

$$\lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a)) \neq 0.$$

Hence $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ exists and $f'(a) = 1/g'(f(a))$. So $f'(z) = 1/g'(f(z))$ for $z \in G$. If g is analytic, then g' is continuous, and $g'(f(z))$ is continuous. Therefore, f is analytic. □

Theorem III.2.29

Theorem III.2.29. Let u and v be real-valued functions defined on a region G and suppose u and v have continuous partial derivatives (so we view u and v as functions of x and y where $z = x + iy$). Then $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if the Cauchy-Riemann equations are satisfied.

Proof. (Analytic implies Cauchy-Riemann) Let $f : G \to \mathbb{C}$ be analytic and for $z = x + iy \in G$, $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. We know $f'(z) = \lim_{h\to 0} \frac{f(x+h)-f(z)}{h}$ exists. We consider the limit along two paths. We have for $h \in \mathbb{R}$

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+h+iy)-f(x+iy)}{h}$$

$$= \left(\frac{u(x+h,y)-u(x,y)}{h}\right) + i\left(\frac{v(x+h,y)-v(x,y)}{h}\right)$$

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$$= \left(\frac{u(x + h, y) - u(x, y)}{h}\right) + i\left(\frac{v(x + h, y) - v(x, y)}{h}\right)$$

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Theorem III.2.29 (continued 1)

Proof (continued).

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

Next, let $h \in \mathbb{R}$ and $ih \to 0$. Then

$$\frac{f(z+ih)-f(z)}{ih} = \left(\frac{u(x,y+h)-u(x,y)}{ih}\right) + i\left(\frac{v(x,y+h)-v(x,y)}{ih}\right)$$

$$= -i\left(\frac{u(x,y+h)-u(x,y)}{h}\right) + \left(\frac{v(x,y+h)-v(x,y)}{h}\right)$$

and so $f'(z) = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$. Therefore, $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$. That is, the Cauchy-Riemann equations are necessary. (We have only used *differentiability* here!)

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Theorem III.2.29 (continued 2)

Proof (continued). (Cauchy-Riemann implies analytic) Let G be a region and let u and v be functions defined on G with continuous partial derivatives which satisfy the Cauchy-Riemann equations on G. Let $z = x + iy \in G$ and let $B(z; r) \subset G$. If $h = s + it \in B(0; r)$ then

$$\mathrm{Re}(f(z+h) - f(z)) = u(x+s, y+t) - u(x, y)$$

$$= [u(x+s, y+t) - u(x, y+t)] + [u(x, y+t) - u(x, y)].$$

Treating the first bracketed quantity as a function of the first variable and the second bracketed quantity as a function of the second variable, we have by the Mean Value Theorem that for some s_1, t_1 where s_1 is between 0 and s and t_1 is between 0 and t:

$$u_x(x + s_1, y + t) = \frac{u(x+s, y+t) - u(x, y+t)}{s - 0},$$

$$u_y(x, y + t_1) = \frac{u(x, y+t) - u(x, y)}{t - 0}.$$

Theorem III.2.29 (continued 2)

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$$u_y(x, y+t_1) = \frac{u(x, y+t) - u(x,y)}{t-0}.$$

Theorem III.2.29 (continued 3)

Proof (continued). Define

$$\varphi(s,t) = [u(x+s,y+t) - u(x,y)] - [u_x(x,y)s + u_y(x,y)t] \quad (*)$$

and then

$$\frac{\varphi(s,t)}{s+it} = \frac{s}{s+it}[u_x(x+s_1,y+t) - u_x(x,y)] + \frac{t}{s+it}[u_y(x,y+t_1) - u_y(x,y)].$$

Since $|s| \leq |s+it|$ and $|t| \leq |s+it|$, then $\left|\frac{s}{s+it}\right|$ and $\left|\frac{t}{s+it}\right|$ are bounded. Since $|s_1| < |s|$ and $|t_1| < |t|$, and the fact that u_x and u_y are continuous [the continuity of the partials is used here!], then $\lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0$. So by $(*)$,

$$u(x+s,y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t)$$

$$\text{where } \lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0. \quad (**)$$

Theorem III.2.29 (continued 3)

Proof (continued). Define

$$\varphi(s,t) = [u(x+s, y+t) - u(x,y)] - [u_x(x,y)s + u_y(x,y)t] \qquad (*)$$

and then

$$\frac{\varphi(s,t)}{s+it} = \frac{s}{s+it}[u_x(x+s_1, y+t) - u_x(x,y)] + \frac{t}{s+it}[u_y(x, y+t_1) - u_y(x,y)].$$

Since $|s| \le |s+it|$ and $|t| \le |s+it|$, then $\left|\frac{s}{s+it}\right|$ and $\left|\frac{t}{s+it}\right|$ are bounded. Since $|s_1| < |s|$ and $|t_1| < |t|$, and the fact that u_x and u_y are continuous [the continuity of the partials is used here!], then $\lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0$. So by $(*)$,

$$u(x+s, y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t)$$
$$\text{where } \lim_{s+it\to 0} \frac{\varphi(s,t)}{s+it} = 0. \qquad (**)$$

Theorem III.2.29 (continued 4)

Proof (continued). Similarly

$$v(x+s,y+t)-v(x,y)=v_x(x,y)s+v_y(x,y)t+\psi(s,t)$$

$$\text{where } \lim_{s+it\to 0}\frac{\psi(s,t)}{s+it}=0. \qquad (***)$$

Now

$$\frac{f(z+s+it)-f(z)}{s+it}=\frac{\mathrm{Re}(f(z+s+it)-f(z))+i\mathrm{Im}(f(z+s+it)-f(z))}{s+it}$$

$$=\frac{u(x+s,y+t)-u(x,y)+i(v(x+s,y+t)-v(x,y))}{s+it}$$

$$=\frac{(u_x(x,y)s+u_y(x,y)t+\varphi(s,t))+i(v_x(x,y)s+v_y(x,y)t+\psi(s,t))}{s+it}$$

by $(**)$ and $(***)$

$$=\frac{(u_x(x,y)s-v_x(x,y)t)+i(v_x(x,y)s+u_x(x,y)t)}{s+it}+\frac{\varphi(s,t)+i\psi(s,t)}{s+it}$$

by the Cauchy-Riemann equations

Theorem III.2.29 (continued 5)

Proof (continued).

$$= u_x(x,y) + \frac{i^2 v_x(x,y)t + iv_x(x,y)s}{s+it} + \frac{\varphi(s,t) + i\psi(s,t)}{s+it}$$

$$= u_x(x,y) + iv_x(x,y) + \frac{\varphi(s,t) + i\psi(s,t)}{s+it}.$$

With $s + it \to 0$ and since f is differentiable,

$$f'(z) = u_x(x,y) + iv_x(x,y).$$

Since u_x and v_x are continuous, then f' is continuous and so f is analytic (i.e., continuously differentiable). □