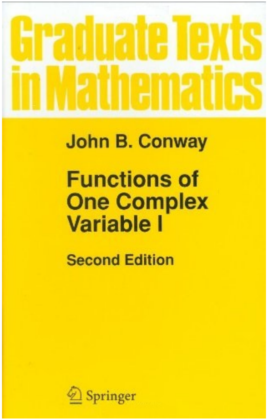
Complex Analysis

Chapter III. Elementary Properties and Examples of Analytic Functions

III.3. Analytic Functions as Mappings, Möbius Transformations—Proofs



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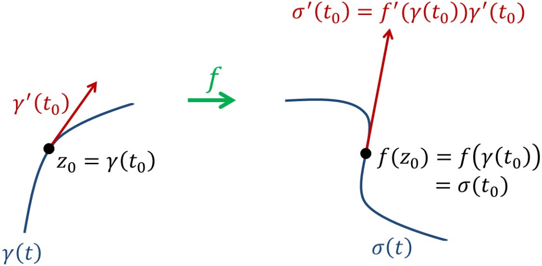

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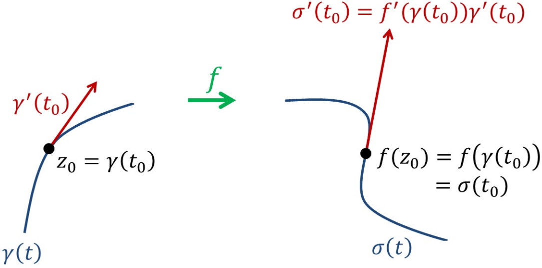

Then $\sigma'(t_0) = f'(\gamma(t_0))\gamma'(t_0) \neq 0$ and

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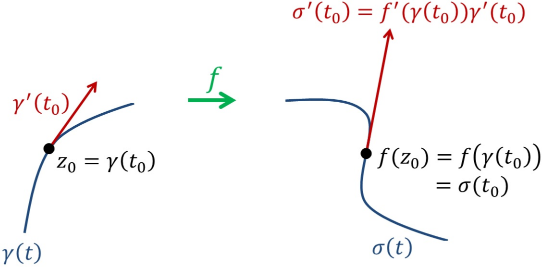

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Proof (continued). So if γ_1 and γ_2 are smooth paths which intersect at x_0 and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$, then $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$ are smooth. So $(*)$ implies

$$\arg(\sigma_1'(t_1)) - \arg(\sigma_2'(t_2)) = \arg(f'(\gamma_1(t_1)) + \arg(\gamma_1'(t_1))$$

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That is, an angle between γ_1 and γ_2 at z_0 is the same as the angle between $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$. □

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Proposition III.3.8. If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct, and T is a Möbius transformation then $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any $z_1 \in \mathbb{C}_\infty$.

Proof. Let $S(z) = (z, z_1, z_2, z_3, z_4)$ (as defined above). Then S is a Möbius transformation. Define $M = S \circ T^{-1}$.

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If R is another Möbius transformation with $Rz_i = \omega_i$ for $i = 2, 3, 4$, then $R^{-1} \circ S$ fixed z_2, z_3, z_4 and so $R^{-1} \circ S = I$, or $R = S$. So the transformation is unique. □

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Proof. Let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined as $S(z) = (z, z_2, z_3, z_4)$. Then $S(z)$ is real if and only if (z, z_2, z_3, z_4) is real. So

$$\{z \mid (z, z_2, z_3, z_4) \in \mathbb{R}\} = \{z \mid S(z) \in \mathbb{R}\} = \{z \mid z \in S^{-1}(\mathbb{R})\}.$$

So we show that the inverse image of $\mathbb{R}_\infty$ is a circle/cline under *any* Möbius transformation. Let $S(z) = \frac{az+b}{cz+d}$. If $z = x \in \mathbb{R}$ and if $\omega = S^{-1} \neq \infty$ (so $x \neq -d/c$) then $x = S(\omega) \in \mathbb{R}$ and so $S(\omega) = \overline{S(\omega)}$. So $\frac{a\omega+b}{c\omega+d} = \frac{\overline{a\omega}+\overline{b}}{\overline{c\omega}+\overline{d}}$. Therefore $(a\omega + b)(\overline{c\omega} + \overline{d}) = (c\omega + d)(\overline{a\omega} + \overline{b})$ or $a\overline{c}|\omega|^2 + a\overline{d}\omega + b\overline{c\omega} + b\overline{d} = \overline{a}c|\omega|^2 + d\overline{a\omega} + c\overline{b}\omega + d\overline{b}$ or

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$$\delta = \frac{\overline{b}d - b\overline{d}}{a\overline{c} - \overline{a}c} = \frac{i2\mathrm{Im}(\overline{b}d)}{i2\mathrm{Im}(a\overline{c})} = \frac{\mathrm{Im}(\overline{b}d)}{\mathrm{Im}(a\overline{c})} \in \mathbb{R}.$$

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Hence

$$|\omega + \gamma| = \sqrt{|\gamma|^2 + \delta} = \sqrt{\underbrace{\frac{b\overline{c} - d\overline{a}}{a\overline{c} - \overline{a}c}}_{\gamma}\underbrace{\frac{\overline{b}c - \overline{d}a}{\overline{a} - a\overline{c}}}_{\overline{\gamma}} + \underbrace{\frac{\overline{b}d - b\overline{d}}{a\overline{c} - \overline{a}c}}_{\delta}\underbrace{\frac{\overline{a}c - a\overline{c}}{\overline{a}c - a\overline{c}}}_{1}}$$

$$= \frac{1}{|a\overline{c} - \overline{a}c|}\{b\overline{c}\overline{b}c - b\overline{c}\overline{d}a - d\overline{a}\overline{b}c + d\overline{a}\overline{d}a + \overline{b}d\overline{a}c - \overline{b}da\overline{c} - b\overline{d}\overline{a}c + b\overline{d}a\overline{c}\}^{1/2}$$

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Proof (continued).
Case 2. Suppose $a\overline{c}$ is not real. Then equation (3.11) becomes $|\omega|^2 + \overline{\gamma}\omega + \gamma\overline{\omega} - \delta = 0$ where $\gamma = \dfrac{b\overline{c} - d\overline{a}}{a\overline{c} - \overline{a}c}$ and $\delta = \dfrac{\overline{b}d - b\overline{d}}{a\overline{c} - \overline{a}c} = \dfrac{i2\text{Im}(\overline{b}d)}{i2\text{Im}(a\overline{c})} = \dfrac{\text{Im}(\overline{b}d)}{\text{Im}(a\overline{c})} \in \mathbb{R}$. So

$|\omega|^2 + \overline{\gamma}\omega + \gamma\overline{\omega} + |\gamma|^2 = |\gamma|^2 + \delta$, or $|\omega + \gamma|^2 = (\omega + \gamma)(\overline{\omega} + \overline{\gamma}) = |\gamma|^2 + \delta$.

Hence

$$|\omega + \gamma| = \sqrt{|\gamma|^2 + \delta} = \sqrt{\underbrace{\frac{b\overline{c} - d\overline{a}}{a\overline{c} - \overline{a}c}}_{\gamma}\underbrace{\frac{\overline{b}c - \overline{d}a}{\overline{a} - a\overline{c}}}_{\overline{\gamma}} + \underbrace{\frac{\overline{b}d - b\overline{d}}{a\overline{c} - \overline{a}c}}_{\delta}\underbrace{\frac{\overline{a}c - a\overline{c}}{\overline{a}c - a\overline{c}}}_{1}}$$

$$= \frac{1}{|a\overline{c} - \overline{a}c|}\left\{b\overline{c}\overline{b}c - b\overline{c}\overline{d}a - d\overline{a}\overline{b}c + d\overline{a}\overline{d}a + \overline{b}d\overline{a}c - \overline{b}da\overline{c} - b\overline{d}\overline{a}c + b\overline{d}a\overline{c}\right\}^{1/2}$$

Proposition III.3.10 (continued 3)

Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle/cline.

Proof (continued).

$$= \frac{1}{|a\overline{c} - \overline{a}c|}\{b\overline{c}\overline{b}c - b\overline{c}\overline{d}a - d\overline{a}\overline{b}c + d\overline{a}\overline{d}a + \overline{b}d\overline{a}c - \overline{b}da\overline{c} - b\overline{d}\overline{a}c + b\overline{d}a\overline{c}\}^{1/2}$$

$$= \frac{1}{|a\overline{c} - \overline{a}c|}\{\overline{b}\overline{c}(bc - ad) - \overline{a}\overline{d}(-ad + bc)\}^{1/2}$$

$$= \frac{1}{|a\overline{c} - \overline{a}c|}\sqrt{(\overline{b}\overline{c} - \overline{a}\overline{d})(bc - ad)} = \frac{|ad - bc|}{|a\overline{c} - \overline{a}c|} > 0$$

since $ad - bc \neq 0$.

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Theorem III.3.14

Theorem III.3.14. A Möbius transformation takes circles/clines onto circles/clines.

Proof. Let Γ be a circle/cline in $\mathbb{C}_\infty$ and let S be a Möbius transformation. Let $z_2.z_3.z_4 \in \mathbb{C}_\infty$ be distinct points on Γ.

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$$(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4) = (Sz, \omega_2, \omega_3, \omega_4)$$

for each $a \in \mathbb{C}_\infty$. Now for each $z \in \Gamma$, (z, z_2, z_3, z_4) is real by Proposition III.3.10.

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If a Möbius transformation takes a circle/cline Γ_1 onto the circle/cline Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

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