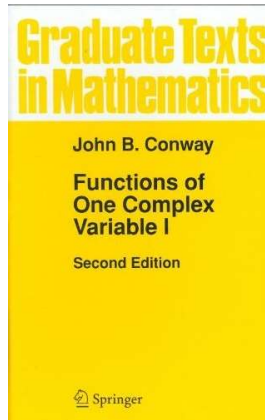


Complex Analysis

Chapter IV. Complex Integration IV.1. Riemann-Stieltjes Integrals—Proofs



Theorem IV.1.3

Proposition IV.1.3. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise smooth then γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Proof. Assume that γ is smooth (the case of piecewise smooth following by summing). Let $P = \{a = t_0 < t_1 < \dots < t_m = b\}$. Then

$$\begin{aligned} v(\gamma; P) &= \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right| \text{ by the FTC since } \gamma \text{ is smooth} \\ &\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt = \int_a^b |\gamma'(t)| dt. \end{aligned}$$

Hence γ is of bounded variation since $V(\gamma) \leq \int_a^b |\gamma'(t)| dt$, (*)

Theorem IV.1.3 (continued 1)

Proof (continued). Since γ' is continuous and $[a, b]$ is compact, then γ' is uniformly continuous. So if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also by definition of integral, there exists $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \dots < t_m = b\}$ and $\|P\| = \max\{t_k - t_{k-1} \mid 1 \leq k \leq m\} < \delta_2$ implies $\left| \int_a^b |\gamma'(t)| dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon$ where τ_k is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &< \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \\ &= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) dt \right| \text{ since } \gamma'(\tau_k) \text{ is constant} \end{aligned}$$

Theorem IV.1.3 (continued 2)

Proof (continued).

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &< \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)] dt \right| \\ &\leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)] dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right|. \end{aligned}$$

If $\|P\| < \delta = \min\{\delta_1, \delta_2\}$ then $|\gamma'(\tau_k) - \gamma'(t)| < \varepsilon$ for $t \in [t_{k-1}, t_k]$ and

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &< \varepsilon + \varepsilon(b - a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \varepsilon[1 + (b - a)] + v(\gamma; P) \leq \varepsilon[1 + b - a] + V(\gamma). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b |\gamma'(t)| dt \leq V(\gamma)$, and we have equality combining with (*). □

Theorem IV.1.4

Theorem IV.1.4. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \rightarrow \mathbb{C}$ is continuous. Then there is a complex number I such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max\{t_k - t_{k-1}\} < \delta$, then

$$\left| I - \sum_{k=1}^m f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of points τ_k , where $\tau_k \in [t_{k-1}, t_k]$. The number I is called the *Riemann-Stieltjes integral* of f with respect to γ over $[a, b]$, denoted

$$I = \int_a^b f d\gamma = \int_a^b f(t) d\gamma(t).$$

Theorem IV.1.4 (continued 2)

Proof (continued). (1) Suppose $P \subset Q$ (and so $Q \in \mathcal{P}_m$) such that $Q = P \cup \{t^*\}$ where $t_{p-1} < t^* < t_p$ (so Q contains one more point than P and is a refinement of P). If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$ and if

$$S(Q) = \sum_{k \neq p} f(\sigma_k)[\gamma(t_k) - \gamma(t_{k-1})] + f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] \\ + f(\sigma')[\gamma(t_p) - \gamma(t^*)]$$

then

$$|S(P) - S(Q)| = \left| \sum_{k \neq p} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right. \\ \left. + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] - S(Q) \right| \\ = \left| \sum_{k \neq p} (f(\tau_k) - f(\sigma_k))[\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] \right. \\ \left. - f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma')[\gamma(t_p) - \gamma(t^*)] \right|$$

Theorem IV.1.4 (continued 1)

Proof. Since f is continuous and $[a, b]$ is compact, then f is uniformly continuous on $[a, b]$. So for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) there exists $\delta_m > 0$ (where we take $\delta_1 > \delta_2 > \delta_3 > \cdots$) such that if $|s - t| < \delta_m$ then $|f(s) - f(t)| < 1/m$. For each $m \in \mathbb{N}$, let \mathcal{P}_m be the set of all partitions P of $[a, b]$ such that $\|P\| < \delta_m$. So $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots$. Define F_m (for each $m \in \mathbb{N}$) as the closure of the set:

$$\left\{ \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \mid P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}. \quad (*)$$

We now show that the diameter of set $(*)$ is $\leq 2/mV(\gamma)$ for each $m \in \mathbb{N}$ for each $m \in \mathbb{N}$. If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$, then denote by $S(P)$ a sum of the form $\sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]$ where τ_k is any point with $t_{k-1} \leq \tau_k \leq t_k$. Fix $m \in \mathbb{N}$ and let $P \in \mathcal{P}_m$.

Theorem IV.1.4 (continued 3)

Proof (continued).

$$\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + |[f(\tau_p) - f(\sigma)][\gamma(t^*) - \gamma(t_{p-1})]| \\ + |[f(\tau_p) - f(\sigma')][\gamma(t_p) - \gamma(t^*)]| \text{ (since } |\tau_k - \sigma_k| < \delta_m \\ \text{and so } f(\tau_k) - f(\sigma_k) < 1/m) \\ \leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| + \frac{1}{m} |\gamma(t_p) - \gamma(t^*)| \\ \leq \frac{1}{m} V(\gamma) \text{ (since } t^* - t_{p-1} < \delta_m \text{ and } |t_p - t^*| < \delta_m).$$

Now if $P \subset Q$ and Q contains several more points than P , then the proof follows similarly.

Theorem IV.1.4 (continued 4)

Proof (continued). Now let P and R be any two partitions in \mathcal{P}_m . Then $Q = P \cup R$ is a refinement of both P and R . By the above argument,

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

Therefore, the modulus of the difference of any two elements of set (*) is $\leq \frac{1}{m} V(\gamma)$. That is, the diameter of set (*) is $\leq \frac{2}{m} V(\gamma)$ and so $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$. So the sets F_m are closed, nested ($F_1 \supset F_2 \supset F_3 \supset \dots$), and $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$ (and so $\text{diam}(F_m) \rightarrow 0$ as $m \rightarrow \infty$). Therefore by Cantor's Theorem (Theorem II.3.7), $\bigcap_{m=1}^{\infty} F_m = \{I\}$ for some single $I \in \mathbb{C}$. This value I satisfies the claims of the theorem. \square

Theorem IV.1.9

Theorem IV.1.9. If γ is piecewise smooth and $f : [a, b] \rightarrow \mathbb{C}$ is continuous then

$$\int_a^b f d\gamma = \int_a^b f(t)\gamma'(t) dt.$$

Proof. Without loss of generality, γ is smooth (the result for piecewise smooth following then from additivity). Also, γ can be represented as $\gamma = \gamma_r + i\gamma_i$ where γ_r and γ_i are real. So also WLOG, $\gamma([a, b]) \subset \mathbb{R}$ (the general result following for complex valued γ by linearity). Let $\varepsilon > 0$ and choose $\delta > 0$ such that if $P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$ satisfies $\|P\| < \delta$ then

$$\left| \int_a^b f d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2} \quad (1.10)$$

$$\text{and } \left| \int_a^b f(t)\gamma'(t) dt - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2} \quad (1.11)$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \dots, n$.

Theorem IV.1.9 (continued)

Theorem IV.1.9. If γ is piecewise smooth and $f : [a, b] \rightarrow \mathbb{C}$ is continuous then

$$\int_a^b f d\gamma = \int_a^b f(t)\gamma'(t) dt.$$

Proof (continued). By the Mean Value Theorem (for real functions from Calculus 1) there is $\tau_k \in [t_{k-1}, t_k]$ with $\gamma'(\tau_k) = [\gamma(t_k) - \gamma(t_{k-1})]/(t_k - t_{k-1})$. Thus

$$\sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] = \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}).$$
 Therefore

$$\left| \int_a^b f d\gamma - \int_a^b f(t)\gamma'(t) dt \right| = \left| \int_a^b f d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right|$$

$$+ \left| \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) - \int_a^b f(t)\gamma'(t) dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by (1.10) and (1.11). \square

Proposition IV.1.13

Proposition IV.1.13. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \rightarrow [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any f continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_{\gamma} f = \int_{\gamma \circ \sigma} f$.

Proof. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that for $P_1 = \{c = s_0 < s_1 < \dots < s_n = d\}$ a partition of $[c, d]$ with $\|P_1\| < \delta_1$ and $s_{k-1} \leq \sigma_k \leq s_k$ we have

$$\left| \int_{\gamma \circ \sigma} f - \sum_{k=1}^n f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.$$

Choose $\delta_2 > 0$ such that if $P_2 = \{a = t_0 < t_1 < \dots < t_n = b\}$ is a partition of $[a, b]$ with $\|P_2\| < \delta_2$ and $t_{k-1} < \tau_k < t_k$ then

$$\left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2}.$$

Proposition IV.1.13 (continued)

Proposition IV.1.13. If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \rightarrow [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any f continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_{\gamma} f = \int_{\gamma \circ \sigma} f$.

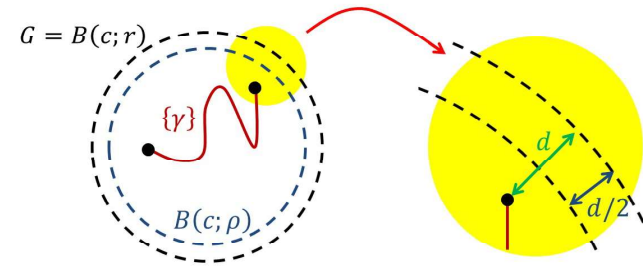
Proof (continued). Since φ is continuous on $[c, d]$ and $[c, d]$ is compact, then there is a $\delta > 0$ such that $\delta < \delta_1$ and $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$ (by the definition of uniform continuity). So if $P_2 = \{c = s_0 < s_1 < \dots < s_n = d\}$ is a partition of $[c, d]$ with $\|P_2\| < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $P_4 = \{a = t_0 \leq t_1 \leq \dots \leq t_n = b\}$ is a partition of $[a, b]$ with $\|P_4\| < \delta_2$. If $s_{k-1} \leq \sigma_k \leq s_k$ and $\tau_k = \varphi(\sigma_k)$ then both above inequalities hold and $|\int_{\gamma} f - \int_{\gamma \circ \sigma} f| = |\int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] + \sum_{k=1}^n f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \varphi(s_k) - \gamma \circ \varphi(s_{k-1})] - \int_{\gamma \circ \sigma} f| < \varepsilon$ and the result follows. \square

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Lemma IV.1.19

Lemma IV.1.19. If G is an open set in \mathbb{C} , $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $|\int_{\gamma} f - \int_{\Gamma} f| < \varepsilon$.

Proof. Case I. Suppose G is an open disk. Since $\{\gamma\}$ is a compact set, by Theorem II.5.17, $d = \text{dist}\{\gamma\}, \partial(G) > 0$ where $\partial(G)$ is the boundary of G . So if $G = B(c; r)$ then $\{\gamma\} \subset B(c; \rho)$ where $\rho = r - \frac{1}{2}d$:



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Lemma IV.1.19 (continued 1)

Proof (continued). Case I (continued 1). Now f is uniformly continuous on $\bar{B}(c; \rho) \subset G$ since $\bar{B}(c; \rho)$ is compact. So WLOG, f is uniformly continuous on G . Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. γ is defined on $[a, b]$ and so γ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \dots < t_n = b\}$ of $[a, b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for s, t such that $t_{k-1} \leq s \leq t_k$ and $t_{k-1} \leq t \leq t_k$, and (2) for $\tau_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon \quad (1.20)$$

(by the definition of $\int_{\gamma} f$). We now use this partition of $[a, b]$ to define the desired polygon. Define $\Gamma : [a, b] \rightarrow \mathbb{C}$ as

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}} [(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)] \text{ for } t \in [t_{k-1}, t_k].$$

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Lemma IV.1.19 (continued 2)

Proof (continued). Case I. (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}), \gamma(t_k)]$). Then Γ is a polygonal path and a subset of G (since G is convex; it's a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \leq s \leq t \leq t_k$, then

$$\begin{aligned} |\Gamma(t) - \gamma(\tau_k)| &= |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)| \\ &\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21) \end{aligned}$$

for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$).



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Lemma IV.1.19 (continued 3)

Proof (continued). Case I. Since $\int_{\Gamma} f = \int_a^b f(\Gamma(t))\Gamma'(t) dt$ (computed piecewise), then

$$\int_{\Gamma} f = \sum_{k=1}^n \underbrace{\left(\frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right)}_{\Gamma'(t)} \int_{t_{k-1}}^{t_k} f(\Gamma(t)) dt.$$

Next,

$$\begin{aligned} \left| \int_{\gamma} f - \int_{\Gamma} f \right| &= \left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right. \\ &\quad \left. + \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \right| \\ &< \varepsilon + \left| \sum_{k=1}^m f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \right| \text{ by (1.20)} \end{aligned}$$

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Lemma IV.1.19 (continued 4)

Proof (continued). Case I.

$$\begin{aligned} \left| \int_{\gamma} f - \int_{\Gamma} f \right| &= \varepsilon + \left| \sum_{k=1}^n (f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right. \\ &\quad \left. - \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t)) dt \right) \Big| \\ &= \varepsilon + \left| \sum_{k=1}^n \left(\frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right) \int_{t_{k-1}}^{t_k} \underbrace{(f(\gamma(\tau_k)) - f(\Gamma(t)))}_{\text{constant WRT } t} dt \right| \\ &\leq \varepsilon + \sum_{k=1}^n \left(\frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))| dt \right). \end{aligned}$$

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Lemma IV.1.19 (continued 5)

Lemma IV.1.19. If G is an open set in \mathbb{C} , $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

Proof (continued). Case I. To recap: G is an open disk and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| \leq \varepsilon + \sum_{k=1}^n \left(\frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))| dt \right).$$

By (1.21), $|\Gamma(t) - \gamma(\tau_k)| < \delta$ and by uniform continuity mentioned above, $|f(\gamma(\tau_k)) - f(\Gamma(t))| < \varepsilon$, so

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon = \varepsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \leq \varepsilon(1 - V(\gamma)).$$

Since ε is arbitrary, Case I follows.

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Lemma IV.1.19 (continued 6)

Lemma IV.1.19. If G is an open set in \mathbb{C} , $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

Proof (continued). Case II. G is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number r such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of γ on $[a, b]$). If $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the “ k th part” of γ contained in $B(\gamma(t_{k-1}); r)$ and can use Case I. If $\gamma_k : [t_{k-1}, t_k] \rightarrow G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \leq k \leq n$ (the “parts” of γ). By Case I there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \rightarrow B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. Defining Γ as the union of the Γ_k yields the desired polygonal path. \square

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Theorem IV.1.18

Theorem IV.1.18. Let G be open in \mathbb{C} and let γ be a rectifiable path in G with initial and end points α and β . If $f : G \rightarrow \mathbb{C}$ is a continuous function with a *primitive* $F : G \rightarrow \mathbb{C}$ (i.e., $F' = f$), then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

Proof. Case I. Suppose $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise smooth. Then

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(\gamma(t))\gamma'(t) dt \text{ (piecewise)} \\ &= \int_a^b F'(\gamma(t))\gamma'(t) dt = \int_a^b (F \circ \gamma)'(t) dt \\ &= \int_a^b \operatorname{Re}\{(F \circ \gamma)'\} dt + i \int_a^b \operatorname{Im}\{(F \circ \gamma)'\} dt \\ &= \operatorname{Re}\{(F \circ \gamma)\}' \Big|_a^b + i \operatorname{Im}\{(F \circ \gamma)\}' \Big|_a^b \text{ by the F.T.C.} \\ &= F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

Theorem IV.1.18 (continued)

Theorem IV.1.18. Let G be open in \mathbb{C} and let γ be a rectifiable path in G with initial and end points α and β . If $f : G \rightarrow \mathbb{C}$ is a continuous function with a *primitive* $F : G \rightarrow \mathbb{C}$ (i.e., $F' = f$), then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

Proof (continued). Case II. Suppose γ is rectifiable. For $\varepsilon > 0$, Lemma IV.1.19 implies there is a polygonal path Γ from α to β such that $|\int_{\gamma} f - \int_{\Gamma} f| < \varepsilon$. But Γ is piecewise smooth, so by Case I, $\int_{\Gamma} f = F(\beta) - F(\alpha)$. Therefore $|\int_{\gamma} f - [F(\beta) - F(\alpha)]| < \varepsilon$, and the result follows. \square