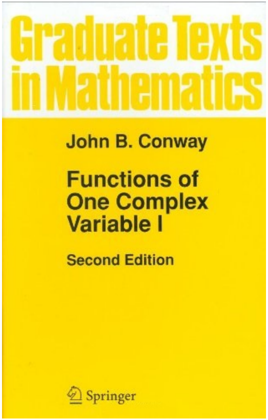
# Complex Analysis

## Chapter IV. Complex Integration

IV.1. Riemann-Stieltjes Integrals—Proofs



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# Theorem IV.1.3

**Proposition IV.1.3.** If $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth then $\gamma$ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

**Proof.** Assume that $\gamma$ is smooth (the case of piecewise smooth following by summing). Let $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$. Then

$$\begin{aligned} v(\gamma; P) &= \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\, dt \right| \text{ by the FTC since } \gamma \text{ is smooth} \\ &\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} |\gamma'(t)|\, dt = \int_a^b |\gamma'(t)|\, dt. \end{aligned}$$

Hence $\gamma$ is of bounded variation since $V(\gamma) \leq \int_a^b |\gamma'(t)|\, dt$, $(*)$

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Hence $\gamma$ is of bounded variation since $\quad V(\gamma) \leq \int_a^b |\gamma'(t)|\, dt, \quad (*)$

# Theorem IV.1.3 (continued 1)

**Proof (continued).** Since $\gamma'$ is continuous and $[a, b]$ is compact, then $\gamma'$ is uniformly continuous. So if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also by definition of integral, there exists $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and $\|P\| = \max\{t_k - t_{k-1} \mid 1 \leq k \leq m\} < \delta_2$ implies $\left| \int_a^b |\gamma'(t)|\, dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon$ where $\tau_k$ is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned} \int_a^b |\gamma'(t)|\, dt &< \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \\ &= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k)\, dt \right| \text{ since } \gamma'(\tau_k) \text{ is constant} \end{aligned}$$

# Theorem IV.1.3 (continued 1)

**Proof (continued).** Since $\gamma'$ is continuous and $[a, b]$ is compact, then $\gamma'$ is uniformly continuous. So if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also by definition of integral, there exists $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and $\|P\| = \max\{t_k - t_{k-1} \mid 1 \le k \le m\} < \delta_2$ implies $\left| \int_a^b |\gamma'(t)|\, dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon$ where $\tau_k$ is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned} \int_a^b |\gamma'(t)|\, dt &< \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \\ &= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k)\, dt \right| \text{ since } \gamma'(\tau_k) \text{ is constant} \end{aligned}$$

# Theorem IV.1.3 (continued 2)

**Proof (continued).**

$$\begin{aligned}\int_a^b |\gamma'(t)|\,dt &< \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)]\,dt \right| \\ &\le \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)]\,dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\,dt \right|.\end{aligned}$$

If $\|P\| < \delta = \min\{\delta_1, \delta_2\}$ then $|\gamma'(\tau_k) - \gamma'(t)| < \varepsilon$ for $t \in [t_{k-1}, t_k]$ and

$$\int_a^b |\gamma'(t)|\,dt < \varepsilon + \varepsilon(b-a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|$$

$$= \varepsilon[1 + (b-a)] + v(\gamma; P) \le \varepsilon[1 + b - a] + V(\gamma).$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b |\gamma'(t)|\,dt \le V(\gamma)$, and we have equality combining with $(*)$. □

# Theorem IV.1.3 (continued 2)

**Proof (continued).**

$$\begin{aligned}\int_a^b |\gamma'(t)|\,dt &< \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)]\,dt \right| \\ &\le \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)]\,dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\,dt \right|.\end{aligned}$$

If $\|P\| < \delta = \min\{\delta_1, \delta_2\}$ then $|\gamma'(\tau_k) - \gamma'(t)| < \varepsilon$ for $t \in [t_{k-1}, t_k]$ and

$$\int_a^b |\gamma'(t)|\,dt < \varepsilon + \varepsilon(b-a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|$$

$$= \varepsilon[1 + (b-a)] + v(\gamma; P) \le \varepsilon[1 + b - a] + V(\gamma).$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b |\gamma'(t)|\,dt \le V(\gamma)$, and we have equality combining with $(*)$. □

# Theorem IV.1.4

**Theorem IV.1.4.** Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then there is a complex number $I$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max\{t_k - t_{k-1}\} < \delta$, then

$$\left| I - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of points $\tau_k$, where $\tau_k \in [t_{k-1}, t_k]$. The number $I$ is called the *Riemann-Stieltjes integral* of $f$ with respect to $\gamma$ over $[a, b]$, denoted

$$I = \int_a^b f \, d\gamma = \int_a^b f(t) \, d\gamma(t).$$

# Theorem IV.1.4 (continued 1)

**Proof.** Since $f$ is continuous and $[a, b]$ is compact, then $f$ is uniformly continuous on $[a, b]$. So for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) there exists $\delta_m > 0$ (where we take $\delta_1 > \delta_2 > \delta_3 > \cdots$) such that if $|s - t| < \delta_m$ then $|f(s) - f(t)| < 1/m$. For each $m \in \mathbb{N}$, let $\mathcal{P}_m$ be the set of all partitions $P$ of $[a, b]$ such that $\|P\| < \delta_m$. So $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots$. Define $F_m$ (for each $m \in \mathbb{N}$) as the closure of the set:

$$\left\{ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \,\middle|\, P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}. \qquad (*)$$

We now show that the diameter of set $(*)$ is $\leq 2/mV(\gamma)$ for each $m \in \mathbb{N}$ for each $m \in \mathbb{N}$. If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$, then denote by $S(P)$ a sum of the form $\sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]$ where $\tau_k$ is any point with $t_{k-1} \leq \tau_k \leq t_k$. Fix $m \in \mathbb{N}$ and let $P \in \mathcal{P}_m$.

# Theorem IV.1.4 (continued 1)

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$$\left\{ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \,\middle|\, P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}. \qquad (*)$$

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# Theorem IV.1.4 (continued 2)

**Proof (continued).** (1) Suppose $P \subset Q$ (and so $Q \in \mathcal{P}_m$) such that $Q = P \cup \{t^*\}$ where $t_{p-1} < t^* < t_p$ (so $Q$ contains one more point than $P$ and is a refinement of $P$). If $t_{p-1} \le \sigma \le t^*$ and $t^* \le \sigma' \le t_p$ and if

$$S(Q) = \sum_{k \neq p} f(\sigma_k)[\gamma(t_k) - \gamma(t_{k-1})] + f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})]$$

$$+f(\sigma')[\gamma(t_p) - \gamma(t^*)]$$

then

$$|S(P) - S(Q)| = |\sum_{k \neq p} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]$$

$$+f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] - S(Q)|$$

$$= |\sum_{k \neq p}(f(\tau_k) - f(\sigma_k))[\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})]$$

$$-f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma')[\gamma(t_p) - \gamma(t^*)]|$$

# Theorem IV.1.4 (continued 3)

**Proof (continued).**

$$\leq \frac{1}{m}\sum_{k\neq p}|\gamma(t_k)-\gamma)t_{k-1})| + |[f(\tau_p)-f(\sigma)][\gamma(t^*)-\gamma(t_{p-1})]$$

$$+[f(\tau_p)-f(\sigma')][\gamma(t_p)-\gamma(t^*)]| \text{ (since } |\tau_k-\sigma_k|<\delta_m$$

$$\text{and so } f(\tau_k)-f(\sigma_k)|<1/m)$$

$$\leq \frac{1}{m}\sum_{k\neq p}|\gamma(t_k)-\gamma(t_{k-1})| + \frac{1}{m}|\gamma(t^*)-\gamma(t_{p-1})| + \frac{1}{m}|\gamma(t_p)-\gamma(t^*)|$$

$$\leq \frac{1}{m}V(\gamma) \text{ (since } t^*-t_{p-1}|<\delta_m \text{ and } |t_p-t^*|<\delta_m).$$

Now if $P\subset Q$ and $Q$ contains several more points than $P$, then the proof follows similarly.

# Theorem IV.1.4 (continued 4)

**Proof (continued).** Now let $P$ and $R$ be any two partitions in $\mathcal{P}_m$. Then $Q = P \cup R$ is a refinement of both $P$ and $R$. By the above argument,

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

Therefore, the modulus of the difference of any two elements of set $(*)$ is $\leq \frac{1}{m} V(\gamma)$. That is, the diameter of set $(*)$ is $\leq \frac{2}{m} V(\gamma)$ and so $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$. So the sets $F_m$ are closed, nested $(F_1 \supset F_2 \supset F_2 \supset \cdots)$, and $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$ (and so $\text{diam}(F_m) \to 0$ as $m \to \infty$). Therefore by Cantor's Theorem (Theorem II.3.7), $\cap_{m=1}^{\infty} F_m = \{I\}$ for some single $I \in \mathbb{C}$. This value $I$ satisfies the claims of the theorem. □

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# Theorem IV.1.9

**Theorem IV.1.9.** If $\gamma$ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous then

$$\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)\, dt.$$

**Proof.** Without loss of generality, $\gamma$ is smooth (the result for piecewise smooth following then from additivity). Also, $\gamma$ can be represented as $\gamma = \gamma_r + i\gamma_i$ where $\gamma_r$ and $\gamma_i$ are real. So also WLOG, $\gamma([a, b]) \subset \mathbb{R}$ (the general result following for complex valued $\gamma$ by linearity).

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$$\left| \int_a^b f\, d\gamma - \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2} \qquad (1.10)$$

$$\text{and} \quad \left| \int_a^b f(t)\gamma'(t)\, dt - \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2} \qquad (1.11)$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$.

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$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2} \qquad (1.10)$$

and

$$\left| \int_a^b f(t)\gamma'(t) \, dt - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2} \qquad (1.11)$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$.

# Theorem IV.1.9 (continued)

**Theorem IV.1.9.** If $\gamma$ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous then
$$\int_a^b f\,d\gamma = \int_a^b f(t)\gamma'(t)\,dt.$$

**Proof (continued).** By the Mean Value Theorem (for real functions from Calculus 1) there is $\tau_k \in [t_{k-1}, t_k]$ with $\gamma'(\tau_k) = [\gamma(t_k) - \gamma(t_{k-1})]/(t_k - t_{k-1})$. Thus $\sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] = \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1})$. Therefore

$$\left|\int_a^b f\,d\gamma - \int_a^b f(t)\gamma'(t)\,dt\right| = \left|\int_a^b f\,d\gamma - \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]\right.$$
$$\left. + \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) - \int_a^b f(t)\gamma'(t)\,dt\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by (1.10) and (1.11). □

# Proposition IV.1.13

**Proposition IV.1.13.** If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma\circ\sigma} f$.

**Proof.** Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that for $P_1 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ a partition of $[c, d]$ with $\|P_1\| < \delta_1$ and $s_{k-1} \le \sigma_k \le s_k$ we have

$$\left| \int_{\gamma\circ\sigma} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.$$

Choose $\delta_2 > 0$ such that if $P_2 = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P_2\| < \delta_2$ and $t_{k-1} < \tau_k < t_k$ then

$$\left| \int_{\gamma} f - \sum_{k=1}^{n} (\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2}.$$

# Proposition IV.1.13

**Proposition IV.1.13.** If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma\circ\sigma} f$.

**Proof.** Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that for $P_1 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ a partition of $[c, d]$ with $\|P_1\| < \delta_1$ and $s_{k-1} \le \sigma_k \le s_k$ we have

$$\left| \int_{\gamma\circ\sigma} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.$$

Choose $\delta_2 > 0$ such that if $P_2 = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P_2\| < \delta_2$ and $t_{k-1} < \tau_k < t_k$ then

$$\left| \int_\gamma f - \sum_{k=1}^{n} (\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2}.$$

# Proposition IV.1.13 (continued)

**Proposition IV.1.13.** If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma\circ\sigma} f$.

**Proof (continued).** Since $\varphi$ is continuous on $[c, d]$ and $[c, d]$ is compact, then there is a $\delta > 0$ such that $\delta < \delta_1$ and $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$ (by the definition of uniform continuity). So if $P_2 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ is a partition of $[c, d]$ with $\|P_3\| < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $P_4 = \{a = t_0 \le t_1 \le \cdots \le t_n = b\}$ is a partition of $[a, b]$ with $\|P_4\| < \delta_2$. If $s_{k-1} \le \sigma_k \le s_k$ and $\tau_k = \varphi(\sigma_k)$ then both above inequalities hold and

$$\left|\int_\gamma f - \int_{\gamma\circ\sigma} f\right| = \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})]\right.$$

$$\left.+ \sum_{k=1}^n f(\gamma\circ\varphi(\sigma_k))[\gamma\circ\varphi(s_k) - \gamma\circ(\varphi(s_{k-1})] - \int_{\gamma\circ\sigma}\right| < \varepsilon$$ and the result follows. □

# Proposition IV.1.13 (continued)

**Proposition IV.1.13.** If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma\circ\sigma} f$.

**Proof (continued).** Since $\varphi$ is continuous on $[c, d]$ and $[c, d]$ is compact, then there is a $\delta > 0$ such that $\delta < \delta_1$ and $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$ (by the definition of uniform continuity). So if $P_2 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ is a partition of $[c, d]$ with $\|P_3\| < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $P_4 = \{a = t_0 \le t_1 \le \cdots \le t_n = b\}$ is a partition of $[a, b]$ with $\|P_4\| < \delta_2$. If $s_{k-1} \le \sigma_k \le s_k$ and $\tau_k = \varphi(\sigma_k)$ then both above inequalities hold and

$$\left|\int_\gamma f - \int_{\gamma\circ\sigma} f\right| = \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})]\right.$$

$$\left. + \sum_{k=1}^n f(\gamma\circ\varphi(\sigma_k))[\gamma\circ\varphi(s_k) - \gamma\circ(\varphi(s_{k-1})] - \int_{\gamma\circ\sigma}\right| < \varepsilon$$ and the result follows. □

# Lemma IV.1.19

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and
$$\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon.$$

**Proof. Case I.** Suppose $G$ is an open disk. Since $\{\gamma\}$ is a compact set, by Theorem II.5.17, $d = \text{dist})\{\gamma\}, \partial(G)) > 0$ where $\partial(G)$ is the boundary of $G$. So if $G = B(c; r)$ then $\{\gamma\} \subset B(c; \rho)$ where $\rho = r - \frac{1}{2}d$:

# Lemma IV.1.19

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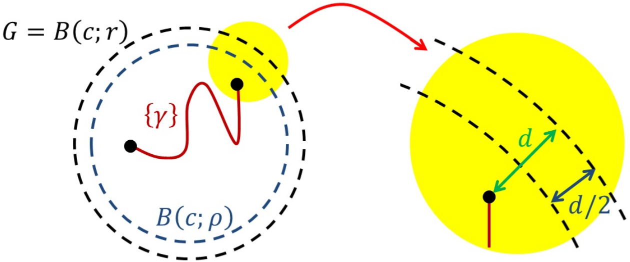

# Lemma IV.1.19

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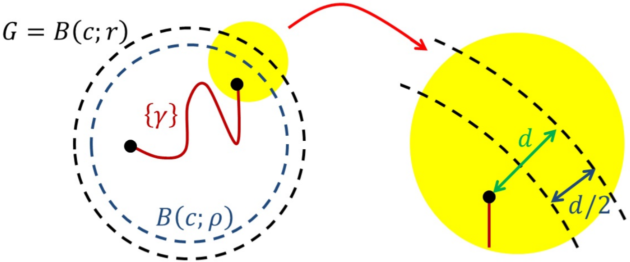

# Lemma IV.1.19 (continued 1)

**Proof (continued). Case I (continued 1).** Now $f$ is uniformly continuous on $\overline{B}(c;\rho) \subset G$ since $\overline{B}(c;\rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a,b]$ and so $\gamma$ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for $s,t$ such that $t_{k-1} \le s \le t_k$ and $t_{k-1} \le t \le t_k$, and (2) for $\tau_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon \qquad (1.20)$$

(by the definition of $\int_\gamma f$). We now use this partition of $[a,b]$ to define the desired polygon. Define $\Gamma : [a,b] \to \mathbb{C}$ as

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}}[(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)] \text{ for } t \in [t_{k-1}, t_k].$$

# Lemma IV.1.19 (continued 1)

**Proof (continued). Case I (continued 1).** Now $f$ is uniformly continuous on $\overline{B}(c;\rho) \subset G$ since $\overline{B}(c;\rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a, b]$ and so $\gamma$ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for $s, t$ such that $t_{k-1} \le s \le t_k$ and $t_{k-1} \le t \le t_k$, and (2) for $\tau_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon \qquad (1.20)$$

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# Lemma IV.1.19 (continued 2)

**Proof (continued). Case I.** (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))]$). Then $\Gamma$ is a polygonal path and a subset of $G$ (since $G$ is convex; it's a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \leq s \leq t \leq t_k$, then

$$|\Gamma(t) - \gamma(\tau_k)| = |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)|$$

$$\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21)$$

for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$:

# Lemma IV.1.19 (continued 2)

**Proof (continued). Case I.** (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))])$. Then $\Gamma$ is a polygonal path and a subset of $G$ (since $G$ is convex; it's a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \le s \le t \le t_k$, then

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for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$:

# Lemma IV.1.19 (continued 2)

**Proof (continued). Case I.** (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))]$). Then $\Gamma$ is a polygonal path and a subset of $G$ (since $G$ is convex; it's a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \leq s \leq t \leq t_k$, then

$$|\Gamma(t) - \gamma(\tau_k)| = |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)|$$

$$\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21)$$

for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$:

# Lemma IV.1.19 (continued 3)

**Proof (continued). Case I.** Since $\int_\Gamma f = \int_a^b f(\Gamma(t))\Gamma'(t)\,dt$ (computed piecewise), then

$$\int_\Gamma f = \sum_{k=1}^{n} \underbrace{\left(\frac{\gamma(t_k)-\gamma(t_{k-1})}{t_k - t_{k-1}}\right)}_{\Gamma'(t)} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,dt.$$

Next,

$$\begin{aligned}\left|\int_\gamma f - \int_\Gamma f\right| &= \left|\int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})]\right.\\ &\quad \left. + \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})] - \int_\Gamma f\right|\\ &< \varepsilon + \left|\sum_{k=1}^{m} f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})] - \int_\Gamma f\right| \text{ by (1.20)}\end{aligned}$$

# Lemma IV.1.19 (continued 4)

**Proof (continued). Case I.**

$$\begin{aligned}
\left|\int_\gamma f - \int_\Gamma f\right| &= \varepsilon + \left|\sum_{k=1}^{n} \left(f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})]\right.\right. \\
&\qquad \left.\left. - \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,dt\right)\right| \\
&= \varepsilon + \left|\sum_{k=1}^{n} \left(\frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}\right) \int_{t_{k-1}}^{t_k} (\underbrace{f(\gamma(\tau_k))}_{\substack{\text{constant}\\ \text{WRT } t}} - f(\Gamma(t)))\,dt\right| \\
&\le \varepsilon + \sum_{k=1}^{n} \left(\frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))|\,dt\right).
\end{aligned}$$

# Lemma IV.1.19 (continued 5)

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$.

**Proof (continued). Case I.** To recap: $G$ is an open disk and

$$\left| \int_\gamma f - \int_\Gamma f \right| \leq \varepsilon + \sum_{k=1}^{n} \left( \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))| \, dt \right).$$

By (1.21), $|\Gamma(t) - \gamma(\tau_k)| < \delta$ and by uniform continuity mentioned above, $|f(\gamma(\tau_k)) - f(\Gamma(t))| < \varepsilon$, so

$$\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon = \varepsilon \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \leq \varepsilon(1 - V(\gamma)).$$

Since $\varepsilon$ is arbitrary, Case I follows.

# Lemma IV.1.19 (continued 6)

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$.

**Proof (continued). Case II.** $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of $\gamma$ on $[a, b]$). If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the "$k$th part" of $\gamma$ contained in $B(\gamma(t_{k-1}; r)$ and can use Case I. If $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \le k \le n$ (the "parts" of $\gamma$). By Case I there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \to B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. Defining $\Gamma$ as the union of the $\Gamma_k$ yields the desired polygonal path. □

# Lemma IV.1.19 (continued 6)

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon.$$

**Proof (continued). Case II.** $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of $\gamma$ on $[a, b]$). If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the "$k$th part" of $\gamma$ contained in $B(\gamma(t_{k-1}; r)$ and can use Case I. If $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \le k \le n$ (the "parts" of $\gamma$). By Case I there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \to B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $| \int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. Defining $\Gamma$ as the union of the $\Gamma_k$ yields the desired polygonal path. □

# Theorem IV.1.18

**Theorem IV.1.18.** Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a *primitive* $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_\gamma f = F(\beta) - F(\alpha)$.

**Proof. Case I.** Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth. Then

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \text{ (piecewise)} \\
&= \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b (F \circ \gamma)'(t)\,dt \\
&= \int_a^b \text{Re}\{(F \circ \gamma)'\}\,dt + i \int_a^b \text{Im}\{(F \circ \gamma)'\}\,dt \\
&= \text{Re}\{(F \circ \gamma)\}|_a^b + i\text{Im}\{(F \circ \gamma)\}|_a^b \text{ by the F.T.C.} \\
&= F(\gamma(b)) - F(\gamma(a)).
\end{aligned}$$

# Theorem IV.1.18 (continued)

**Theorem IV.1.18.** Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a *primitive* $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\displaystyle\int_\gamma f = F(\beta) - F(\alpha)$.

**Proof (continued). Case II.** Suppose $\gamma$ is rectifiable. For $\varepsilon > 0$, Lemma IV.1.19 implies there is a polygonal path $\Gamma$ from $\alpha$ to $\beta$ such that $\left|\int_\gamma f - \int_\Gamma f\right| < \varepsilon$. But $\Gamma$ is piecewise smooth, so by Case I, $\int_\Gamma f = F(\beta) - F(\alpha)$. Therefore $\left|\int_\gamma f - [F(\beta) - F(\alpha)]\right| < \varepsilon$, and the result follows. □