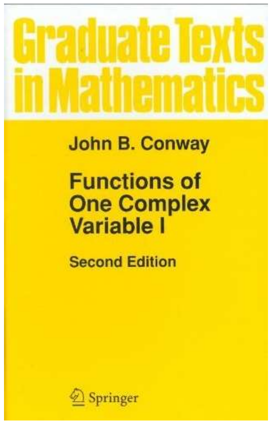
Complex Analysis

Chapter IV. Complex Integration

IV.2. Power Series Representation of Analytic Functions—Proofs



Proposition IV.2.1

Proposition IV.2.1. Let $\varphi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function and define $g : [c, d] \to \mathbb{C}$ by $g(t) = \int_a^b \varphi(s, t)\, ds$. Then g is continuous. Moreover, if $\dfrac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t)\, ds.$$

Proof. The proof that g is continuous is left as Exercise IV.2.1.

Now suppose $\partial\varphi/\partial t$ exists and is continuous on $[a, b] \times [c, d]$. Since $[a, b] \times [c, d]$ is a compact subset of $\mathbb{R}^2$ then by Theorem II.5.15, $\partial\varphi/\partial t$ is uniformly continuous on $[a, b] \times [c, d]$. Now denote $\partial\varphi/\partial t = \varphi_2$. Fix a point t_0 is $[c, d]$ and let $\varepsilon > 0$. So there is $\delta > 0$ such that $|\varphi_2(s', t') - \varphi_2(s, t)| < \varepsilon$ whenever $(s - s')^2 + (t - t')^2 < \delta^2$.

Proposition IV.2.1 (continued 1)

Proof (continued). In particular, $|\varphi_2(s, t) - \varphi_2(s, t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$ and $s \in [a, b]$. So for $|t - t_0| < \delta$ and $x \in [a, b]$ we have

$$\left| \int_{t_0}^t (\varphi_2(s, \tau) - \varphi_2(s, t_0))\, d\tau \right| \le \varepsilon |t - t_0|.$$

But for a fixed $s \in [a, b]$, $\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$ is a primitive of $\varphi_2(s, t) - \varphi_2(s, t_0)$, so by the Fundamental Theorem of Calculus we have

$$\begin{aligned} &\left| \int_{t_0}^t (\varphi_2(s, \tau) - \varphi_2(s, t_0))\, d\tau \right| \\ &= |(\varphi(s, t) - t\varphi_2(s, t_0)) - (\varphi(s, t_0) - t_0\varphi_2(s, t_0))| \\ &= |\varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0)| \le \varepsilon |t - t_0| \end{aligned}$$

and this holds for any $s \in [a, b]$ when $|t - t_0| < \delta$.

Proposition IV.2.1 (continued 2)

Proof (continued). Therefore for $s \in [a, b]$ and $|t - t_0| < \delta$ we have

$$\left| \frac{\varphi(s, t) - \varphi(s, t_0)}{t - t_0} - \varphi_2(s, t_0) \right| \le \varepsilon \text{ and}$$

$$\left| \int_a^b \frac{\varphi(s, t) - \varphi(s, t_0)}{t - t_0}\, ds - \int_a^b \varphi_2(s, t_0)\, ds \right| \le \varepsilon(b - a) \text{ or}$$

$$\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s, t_0)\, ds \right| \le \varepsilon(b - a)$$

since $g(t) = \int_a^b \varphi(s, t)\, ds$ by definition. Therefore for $s \in [a, b]$ we have

$$g'(t_0) = \int_a^b \varphi_2(s, t_0)\, ds = \int_a^b \frac{\partial \varphi}{\partial t}(s, t_0)\, ds.$$

Proposition IV.2.1 (continued 3)

Proposition IV.2.1. Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and define $g : [c,d] \to \mathbb{C}$ by $g(t) = \int_a^b \varphi(s,t)\,ds$. Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b] \times [c,d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\,ds.$$

Proof (continued). Since t_0 is an arbitrary element of $[c,d]$ then we have $g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\,ds$ on $[a,b] \times [c,d]$, as claimed. Since $\partial\varphi/\partial t$ is hypothesized to be continuous then g' is continuous by Exercise IV.2.1 (with g and φ of the exercise replaced with g' and $\partial\varphi/\partial t$ here), as claimed. □

Lemma IV.2.A

Lemma IV.2.A. If $|z| < 1$ then $\int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\,ds = 2\pi$.

Proof. Let $\varphi(s,t) = \frac{e^{is}}{e^{is} - tz}$ for $0 \le t \le 1$ and $0 \le s \le 2\pi$. Since $|z| < 1$, φ is continuously differentiable. So by Proposition IV.2.1, $g(t) = \int_0^{2\pi} \varphi(s,t)\,ds$ is continuously differentiable. Also,

$$g(0) = \int_0^{2\pi} \varphi(s,0)\,ds = \int_0^{2\pi} \frac{e^{is}}{e^{is} - 0z}\,dz = \int_0^{2\pi} 1\,dz = 2\pi.$$

Next, $g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2}\,ds$ by Proposition IV.2.1. Notice for $\Phi(s) = \frac{zi}{e^{is} - tz}$ (with t fixed) we have $\Phi'(s) = \frac{ze^{is}}{(e^{is}-tz)^2}$ and so $\Phi(s)$ is a primitive for $\frac{ze^{is}}{(e^{is} - tz)^2}$, and so

Lemma IV.2.A (continued)

Lemma IV.2.A. If $|z| < 1$ then $\int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\,ds = 2\pi$.

Proof (continued).

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2}\,ds = \Phi(2\pi) - \Phi(0) = \frac{zi}{e^{2\pi i} - tz} - \frac{z}{e^0 - tz} = 0.$$

Therefore g is constant and $g(1) = g(0) = 2\pi$. That is,

$$g(1) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\,dz = 2\pi.$$

□

Theorem IV.2.6

Proposition IV.2.6. Let $f : G \to \mathbb{C}$ be analytic and suppose $\overline{B}(a;r) \subseteq G$ $(r > 0)$. If $\gamma(t) = a + re^{it}$, and $0 \le t \le 2\pi$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\,dw$$

for $|z - a| < r$.

Proof. Without loss of generality, we assume $a = 0$ and $r = 1$ (otherwise, we consider $g(z) = f(a + rz)$ and $G_1 = \{\frac{1}{r}(z - a) \mid z \in G\}$). That is, $\overline{B}(0,1) \subset G$. Fix z where $|z| < 1$. We then need to show that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\,dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z}\,ds.$$

This is equivalent to

$$0 = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z}\,ds - 2\pi f(z) = \int_0^{2\pi} \left(\frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) ds. \quad (*)$$

Theorem IV.2.6 (continued 1)

Proof (continued). Let $\varphi(s,t) = \dfrac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z} - f(z)$ for $0 \le t \le 1$ and $0 \le s \le 2\pi$. Since $|z+t(e^{is}-z)| = |z(1-t)+te^{is}| \le |z(1-t)|+t \le |1-t|+t = 1-t+t = 1$, then φ is well defined (f takes on values in $\overline{B}(0;1) \subset G$) and is continuously differentiable. Let $g(t) = \int_0^{2\pi} \varphi(s,t)\,ds$. Then by Proposition IV.2.1, g is continuously differentiable. Notice that

$$\begin{aligned} g(0) &= \int_0^{2\pi} \varphi(s,0)\,ds = \int_0^{2\pi} \left(\frac{f(z)e^{is}}{e^{is}-z} - f(z) \right) ds \\ &= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,ds - 2\pi f(z) \\ &= 0 \text{ by Lemma IV.2.A} \end{aligned}$$

We now show g is constant. By Proposition IV.2.1, $g'(t) = \int_0^{2\pi} \varphi_2(s,t)\,ds$ where $\varphi_2(s,t) = e^{is} f'(z+t(e^{is}-z)) = \partial\varphi/\partial t$.

Theorem IV.2.6 (continued 2)

Proof (continued). For $0 < t \le 1$, we have $\Phi(s) = -it^{-1} f(z+t(e^{is}-z))$ is a primitive of $\varphi_2(s,t)$. So $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ for $0 < t \le 1$. Since g' is continuous, we must have $g'(t) = 0$ for $0 \le t \le 1$. Therefore $g(t)$ is constant on $[0,1]$ and $g(1) = g(0) = 0$. That is,

$$\begin{aligned} g(1) = \int_0^{2\pi} \varphi(s,1)\,ds &= \int_0^{2\pi} \left(\frac{f(z+1(e^{is}-z))e^{is}}{e^{is}-z} - f(z) \right) ds \\ &= \int_0^{2\pi} \left(\frac{f(e^{is})e^{is}}{e^{is}-z} - f(z) \right) ds = 0. \end{aligned}$$

This is $(*)$ and the result follows. □

Lemma IV.2.7

Lemma IV.2.7. Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous on $\{\gamma\}$ If F is the uniform limit of F_n on $\{\gamma\}$ then $\displaystyle\int_\gamma F = \lim \left(\int_\gamma F_n \right)$.

Proof. Let $\varepsilon > 0$; then there is $N \in \mathbb{N}$ such that $|F_n(w) - F(w)| < \varepsilon / V(\gamma)$ for all $w \in \{\gamma\}$ and $n \ge N$. Then

$$\begin{aligned} \left| \int_\gamma F - \int_\gamma F_n \right| &= \left| \int_\gamma (F - F_n) \right| \\ &\le \int_\gamma |F(w) - F_n(w)|\,|dw| \text{by Proposition IV.1.17} \\ &< \frac{\varepsilon}{V(\gamma)} V(\gamma) = \varepsilon \end{aligned}$$

for all $n \ge N$. So $\int_\gamma F = \lim(\int_\gamma F_n)$. □

Theorem IV.2.8

Theorem IV.2.8. Let f be analytic in $B(a;R)$. Then $f(z) = \displaystyle\sum_{n=0}^\infty a_n (z-a)^n$ for $|z-a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\ge R$.

Proof. Let $0 < r < R$ and then $\overline{B}(a;r) \subset B(a;R)$. If $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, then by Proposition IV.2.6, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{w-z}\,dw$ for $|z-a| < r$. For $|z-a| < r$ and $w \in \{\gamma\}$, $\dfrac{|f(w)||z-a|^n}{|w-a|^{n+1}} \le \dfrac{M}{r} \left(\dfrac{|z-a|}{r} \right)^n$ where $M = \max\{|f(w)| \mid |w-a| = r\}$. Since $|z-a|/r < 1$, the Weierstrass M-Test (with $M_n = M(|z-a|/r)^n/r$) implies that $\displaystyle\sum_{n=1}^\infty \frac{f(w)(z-a)^n}{(w-a)^{n+1}}$ converges uniformly for $w \in \{\gamma\}$.

Theorem IV.2.8 (continued)

Proof (continued). From Note IV.2.A we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \text{ by Proposition IV.2.6} \\ &= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a}\sum_{n=0}^\infty \left(\frac{z-a}{w-a}\right)^n\right) dw \\ &= \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw\right)(z-a)^n \text{ by Lemma IV.2.7.} \end{aligned}$$

Next set $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw$ and we have $f(z) = \sum_{n=0}^\infty a_n(z-a)^n$ where the series converges if $|z-a| < r$. By Proposition III.2.5, $a_n = f^{(n)}(a)/n!$. So each a_n is (1) independent of z, (2) independent of $\{\gamma\}$, and (3) independent of r. Since r was chosen arbitrarily and $< R$, then the series representation holds for all z such that $|z-a| < R$ and the radius of convergence of the series is at least R. □

Theorem IV.2.14

Theorem IV.2.14. Cauchy's Estimate. Let f be analytic in $B(a;R)$ and suppose $|f(z)| \le M$ for all $z \in B(a;R)$. Then

$$|f^{(n)}(a)| \le \frac{n!M}{R^n}.$$

Proof. By Corollary IV.2.13, for $r < R$ we have

$$\begin{aligned} |f^{(n)}(a)| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw\right| \text{ where } \gamma(t) = a + re^{it}, t \in [0, 2\pi] \\ &\le \frac{n!}{2\pi}\int_\gamma \left|\frac{f(w)}{(w-a)^{n+1}}\right| |dw| \text{ by Proposition IV.1.17(b)} \\ &\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}(2\pi r) \text{ by Proposition IV.1.17(b)} \\ &= \frac{n!M}{r^n}. \end{aligned}$$

Now let $r \to R^-$ and the result follows. □

Proposition IV.2.15

Proposition IV.2.15. Let f be analytic in $B(a;R)$ and suppose γ is a closed rectifiable curve in $B(a;R)$. Then f has a primitive in $B(a;R)$ and so $\int_\gamma f = 0$.

Proof. We know by Theorem IV.2.8, that an analytic function has a power series representation: $f(z) = \sum_{n=0}^\infty a_n(z-a)^n$ for $z \in B(a:R)$. Define

$$F(z) = \sum_{n=0}^\infty \frac{a_n}{n+1}(z-a)^{n+1} = (z-a)\sum_{n=0}^\infty \frac{a_n}{n+1}(z-a)^n.$$

Then, by definition, the radius of convergence of F is

$$\frac{1}{\overline{\lim}\left|\frac{a_n}{n+1}\right|^{1/n}} = \frac{\lim(n+1)^{1/n}}{\overline{\lim}|a_n|^{1/n}} = \frac{1}{\overline{\lim}|a_n|^{1/n}}$$

and so the radius of convergence of F is the same as the radius of convergence of f. So F is defined on $B(a;R)$. Also, by Proposition III.2.5, $F'(z) = f(z)$. So F is a primitive of f and by Corollary IV.1.22, $\int_\gamma f = 0$. □