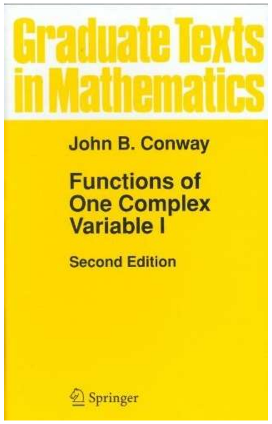
Complex Analysis

Chapter IV. Complex Integration

IV.3. Zeros of an Analytic Functions—Proofs of Theorems



Theorem IV.3.4

Theorem IV.3.4. Liouville's Theorem. If f is a bounded entire function then f is constant.

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Cauchy's Estimate (Corollary IV.2.14) with $n = 1$, $|f'(z)| \leq M/R$ for any disk $B(z; R)$. Since f is entire, the inequality holds for all R and with $R \to \infty$ we see that $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore, f is a constant function by Proposition III.2.10. □

Theorem IV.3.5

Theorem IV.3.5. Fundamental Theorem of Algebra.
If $p(z)$ is a nonconstant polynomial then there is a complex number a with $p(a) = 0$.

Proof. Suppose not. ASSUME $p(z) \neq 0$ for all $z \in \mathbb{C}$. Let $f(z) = 1/p(z)$. Then f is an entire function. If p is not constant, then $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $n \geq 1$ and so

$$\begin{aligned} \lim_{z\to\infty} |p(z)| &= \lim_{z\to\infty} \left| z^n (a_n + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}) \right| \\ &= \lim_{z\to\infty} |z|^n \lim_{z\to\infty} \left| a_n + a_{n-1} z^{-1} + \cdots + a_0 z^{-n} \right| \\ &= \infty. \end{aligned}$$

So $\displaystyle\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{p(z)} = 0$. Therefore, for some $R > 0$ we have $|f(z)| < 1$ for $|z| > R$.

Theorem IV.3.5 (continued)

Theorem IV.3.5. Fundamental Theorem of Algebra.
If $p(z)$ is a nonconstant polynomial then there is a complex number a with $p(a) = 0$.

Proof (continued). Since f is continuous on $\overline{B}(0; R)$ and $\overline{B}(0; R)$ is compact, there is a constant $M > 0$ such that $|f(z)| \leq M$ for $|z| \leq R$ by Corollary II.5.2. Then f is an entire function bounded by $\max\{M, 1\}$. By Liouville's Theorem (Theorem IV.3.4), f must be constant and so p is constant, a CONTRADICTION. So the assumption that $p(z) \neq 0$ for all $z \in \mathbb{C}$ is false and p has some zero in $\mathbb{C}$. □

Theorem IV.3.7

Theorem IV.3.7. Let G be a connected open set and let $f : G \to \mathbb{C}$ be analytic. The following are equivalent.

(a) $f \equiv 0$ on G,

(b) there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{Z}$, $n \geq 0$, and

(c) the set $\{z \in G \mid f(z) = 0\}$ has a limit point in G.

Proof. "Clearly" (a) implies both (b) and (c) (even in the real setting). **(c)⇒(b)** Let $a \in G$ be a limit point of $Z = \{z \in G \mid f(z) = 0\}$ and let $R > 0$ be such that $B(a; R) \subset G$. Since f is analytic, then f is continuous (on G); since a is a limit point of Z then $a = \lim_{n\to\infty}\{z_n\}$ for some sequence $\{z_n\} \subset Z$. So $0 = \lim_{n\to\infty} f(z_n) = f(\lim_{x\to\infty} z_n) = f(a)$. ASSUME there is $n \in \mathbb{N}$ such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$. Since f is analytic, a power series of f is of the form $f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$ for $|z-a| < R$.

Theorem IV.3.7 (continued 1)

Theorem IV.3.7. Let G be a connected open set and let $f : G \to \mathbb{C}$ be analytic. The following are equivalent.

(b) there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{Z}$, $n \geq 0$, and

(c) the set $\{z \in G \mid f(z) = 0\}$ has a limit point in G.

Proof (continued) (c)⇒(b). Define $g(z) = \sum_{k=n}^{\infty} a_k(z-a)^{k-n}$. Then g is analytic in $B(a; R)$, $f(z) = (z-a)^n g(z)$, and $g(a) = a_n \neq 0$ (since $f^{(n)}(a)/n! = a_n \neq 0$). Since g is analytic in $B(a; R)$, we can find r where $0 < r < R$ such that $g(z) \neq 0$ for $|z-a| < r$ (if not, we can repeat the above argument on continuous g to show $g(a) = 0$, a contradiction). Since a is a limit point of G, there is a point b with $f(b) = 0$ and $0 < |b-a| < r$. But then $f(b) = 0 = (b-a)^n g(b)$ and so $g(b) = 0$, CONTRADICTING the fact that g is zero-free in $B(a; r)$. Therefore, the assumption that such an $n \in \mathbb{N}$ exists is false and part (b) holds.

Theorem IV.3.7 (continued 2)

Theorem IV.3.7. Let G be a connected open set and let $f : G \to \mathbb{C}$ be analytic. The following are equivalent.

(a) $f \equiv 0$ on G,

(b) there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{Z}$, $n \geq 0$.

Proof (continued). (b)⇒(a) Let $A = \{z \in G \mid f^{(n)}(z) = 0$ for all $n \geq 0\}$. By hypothesis, $A \neq \varnothing$. [We now show that A is both open in G and closed in G. Since G is connected, then $A = G$; see Definition II.2.1.] (1) Let $a \in \overline{A} \cap G$ and let $\{z_k\} \subset A$ be a sequence such that $\lim\{z_k\} = a$. Since each $f^{(n)}$ is continuous, $f^{(n)}(a) = \lim f^{(n)}(z_k) = 0$ for all $n \geq 0$. So $a \in A$, $A = \overline{A} \cap G$ and A is closed in G. (2) Let $a \in A$ and let $R > 0$ be such that $B(a; R) \subset G$. Then $f(z) = \sum a_n(z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!}f^{(n)}(a) = 0$ for each $n \geq 0$ (by the definition of set A). Hence $f(z) = 0$ for all $z \in B(a; R)$ and so $B(a; R) \subset A$ and A is open in G. Therefore $A = G$, $f^{(n)}(z) = 0$ for all $n \geq 0$ and for all $z \in G$. That is, $f(z) = 0$ on G and (a) holds. □

Corollary IV.3.9

Corollary IV.3.9. If f is analytic on an open connected set G and f is not identically zero then for each $a \in G$ with $f(a) = 0$, there is $n \in \mathbb{N}$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all $z \in G$. That is, each zero of f has finite multiplicity.

Proof. By Theorem IV.3.7, there is a largest $n \in \mathbb{N}$ such that $f^{(k)}(a) = 0$ for $0 \leq k < n$. Define $g(z) = f(z)/(z-a)^n$ for $z \neq a$ and $g(a) = f^{(n)}(a)/n!$. Then g is analytic on $G \setminus \{a\}$. To show g is analytic at $z = a$, we write f as a series $f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$ (as in the proof of Theorem IV.3.7) and then we have $g(z) = \sum_{k=n}^{\infty} a_k(z-a)^{k-n}$ for $z \neq a$, but $a_n = f^{(n)}(a)/n!$ and so g is continuous at $z = a$ and $f(z) = (z-a)^n g(z)$. Since g is written as a series, of course it is analytic. □

Theorem IV.3.11

Theorem IV.3.11. Maximum Modulus Theorem.
If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Proof. Let $\overline{B}(a; r) \subset G$, $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. By Proposition IV.2.6,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)-a}\gamma'(t)\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt.$$

Suppose $|f(a)| \geq |f(z)|$ for all $z \in G$. Then

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\,dt = \frac{1}{2\pi}|f(a)|2\pi = |f(a)|.$$

Theorem IV.3.11 (continued)

Theorem IV.3.11. Maximum Modulus Theorem.
If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Proof (continued). So $|f(a)| = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(a+re^{it})|\,dt$ and

$$0 = 2\pi|f(a)| - \int_0^{2\pi} |f(a+re^{it})|\,dt = \int_0^{2\pi} (|f(a)| - |f(a+re^{it})|\,dt.$$

We've assumed $|f(a)| \geq |f(a+re^{it})|$, so the integrand above is nonnegative. Since the integrand is continuous and the integral is 0, the integrand must be 0. That is, $|f(a)| = |f(a+re^{it})|$ for all $t \in [0, 2\pi]$. Since r is arbitrary, this means any disk $B(a; R) \subset G$ is mapped to something of modulus $|f(a)|$. That is, f maps $B(a; R)$ to $\{z \mid |z| = |f(a)|\}$. By Exercise III.3.17, f is constant on $B(a; R)$, say $f(z) = \alpha$ on $B(a; R)$. By Corollary IV.3.8, $f(z) = \alpha$ for all $z \in G$, and f is constant on G. □