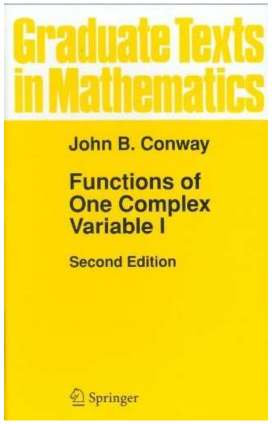
# Complex Analysis

**Chapter IV. Complex Integration**

IV.4. The Index of a Closed Curve—Proofs of Theorems








## Proposition IV.4.1

**Proposition IV.4.1.** If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$

is an integer.

**Proof.** By Lemma IV.1.19, we can approximate $\gamma$ with a polygon $\Gamma$ and then for any $\varepsilon > 0$, we have $\left|\int_\Gamma \frac{1}{z-a}\,dz - \int_\gamma \frac{1}{z-a}\,dz\right| < \varepsilon$. By Proposition IV.1.8, $\int_\Gamma \frac{1}{z-a}\,dz$ can be written as a sum of integrals over the (piecewise smooth) segments of $\Gamma$. So, WLOG, we assume $\gamma$ is smooth. Define $g : [0,1] \to \mathbb{C}$ as $g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\,ds$. Then $g(0) = 0$ and $g(1) = \int_\gamma \frac{1}{z-a}\,dz$.





## Proposition IV.4.1 (continued)

**Proposition IV.4.1.** If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$ is an integer.

**Proof (continued).** Also (by the Fundamental Theorem of Calculus) $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$ for $t \in [0,1]$. But then

$$\frac{d}{dt}[e^{-g(t)}(\gamma(t)-a)] = e^{-g(t)}[\gamma'(t)] + [-g'(t)e^{-g(t)}](\gamma(t)-a)$$

$$= e^{-g(t)}(\gamma'(t) - g'(t)(\gamma(t)-a)) = e^{-g(t)}\left(\gamma'(t) - \frac{\gamma'(t)}{\gamma(t)-a}(\gamma(t)-a)\right) = 0.$$

So $e^{-g(t)}(\gamma(t)-a)$ is constant and $\gamma(0) - a = e^{-g(0)}(\gamma(0)-a) = e^{-g(1)}(\gamma(1)-a)$. Since $\gamma$ is closed, $\gamma(0) = \gamma(1)$ and hence $e^{-g(1)} = e^{-g(0)} = e^0 = 1$, or $g(1) = 2\pi i k$ for some integer $k$. That is, $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz = k \in \mathbb{Z}$. □





## Theorem IV.4.4

**Theorem IV.4.4.** Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for $a$ belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for $a$ belonging to the unbounded component of $G$.

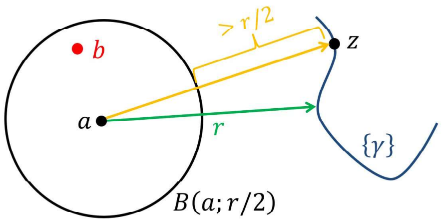
**Proof.** Define $f : G \to \mathbb{C}$ as $f(a) = n(\gamma; a)$. We show that $f$ is continuous (and so constant on the components of $G$ since $n(\gamma; a) \in \mathbb{Z}$). Recall that the components of $G$ are open (by Theorem II.2.9). For fixed $a \in G$, let $r = d(a; \{\gamma\})$. If $|a-b| < \delta < r/2$, then

$$\begin{aligned} |f(a) - f(b)| &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz\right| \text{ by the definition of } n(\gamma; a) \\ &= \frac{1}{2\pi}\left|\int_\gamma \frac{a-b}{(z-a)(z-b)}\,dz\right| \\ &\le \frac{|a-b|}{2\pi}\int_\gamma \frac{|dz|}{|z-a||z-b|}. \end{aligned}$$

# Theorem IV.4.4 (continued 1)

**Theorem IV.4.4.** Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for $a$ belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for $a$ belonging to the unbounded component of $G$.

**Proof (continued).** But for $|a - b| < r/2$ and $z \in \{\gamma\}$, we have $|z - a| \geq r > r/2$ and $|z - b| > r/2$:




Therefore

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi} \int_\gamma \frac{|dz|}{|z-a||z-b|} < \frac{\delta}{2\pi} \frac{2}{r} \frac{2}{r} V(\gamma) = \frac{2\delta}{\pi r^2} V(\gamma)$$

since $1/|z - a| < r/2$ and $1/|z - b| < r/2$.



Theorem IV.4.4

# Theorem IV.4.4 (continued 2)

**Theorem IV.4.4.** Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for $a$ belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for $a$ belonging to the unbounded component of $G$.

**Proof (continued).** So if $\varepsilon > 0$ is given, then $\delta = \min\{r/2, \pi r^2 \varepsilon/(2V(\gamma))\}$ implies that $|f(a) - f(b)| < \varepsilon$. Therefore for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|a - b| < \delta$ then $|f(a) - f(b)| < \varepsilon$. So $f$ is continuous on the components of $G$. That is, $f(a) = n(\gamma; a)$ is constant on the components of $G$.

Now let $U$ be the unbounded component of $G$. Then there is $R > 0$ such that $U \supset \{z \mid |z| > R\}$. If $\varepsilon > 0$ choose $a$ with $|a| > R$ and $|z - a| > \dfrac{V(\gamma)}{2\pi\varepsilon}$ for all $z \in \{\gamma\}$. Then $|n(\gamma; a)| = \left| \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{1}{z-a}\, dz \right|$ $\leq \dfrac{1}{2\pi} V(\gamma) \left( \dfrac{2\pi\varepsilon}{V(\gamma)} \right) = \varepsilon$. Since $\varepsilon$ is arbitrary and $n(\gamma; a)$ is constant for all $a \in G$, then $n(\gamma; a) = 0$ for all $a \in G$. □