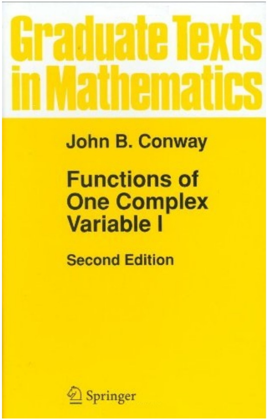
Complex Analysis

Chapter IV. Complex Integration

IV.4. The Index of a Closed Curve—Proofs of Theorems



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$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz$$

is an integer.

Proof. By Lemma IV.1.19, we can approximate γ with a polygon Γ and then for any $\varepsilon > 0$, we have $\left| \int_\Gamma \frac{1}{z-a}\, dz - \int_\gamma \frac{1}{z-a}\, dz \right| < \varepsilon$. By Proposition IV.1.8, $\int_\Gamma \frac{1}{z-a}\, dz$ can be written as a sum of integrals over the (piecewise smooth) segments of Γ. So, WLOG, we assume γ is smooth.

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Proof (continued). Also (by the Fundamental Theorem of Calculus) $g'(t) = \dfrac{\gamma'(t)}{\gamma(t)-a}$ for $t \in [0,1]$. But then

$$\frac{d}{dt}[e^{-g(t)}(\gamma(t)-a)] = e^{-g(t)}[\gamma'(t)] + [-g'(t)e^{-g(t)}](\gamma(t)-a)$$

$$= e^{-g(t)}(\gamma'(t)-g'(t)(\gamma(t)-a)) = e^{-g(t)}\left(\gamma'(t) - \frac{\gamma'(t)}{\gamma(t)-a}(\gamma(t)-a)\right) = 0.$$

So $e^{-g(t)}(\gamma(t)-a)$ is constant and $\gamma(0)-a = e^{-g(0)}(\gamma(0)-a) = e^{-g(1)}(\gamma(1)-a)$. Since γ is closed, $\gamma(0)=\gamma(1)$ and hence $e^{-g(1)} = e^{-g(0)} = e^0 = 1$, or $g(1) = 2\pi i k$ for some integer k. That is, $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z-a}\,dz = k \in \mathbb{Z}$. □

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Theorem IV.4.4

Theorem IV.4.4. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

Proof. Define $f : G \to \mathbb{C}$ as $f(a) = n(\gamma; a)$. We show that f is continuous (and so constant on the components of G since $n(\gamma; a) \in \mathbb{Z}$). Recall that the components of G are open (by Theorem II.2.9).

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$$\begin{aligned} |f(a) - f(b)| &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz\right| \text{ by the definition of } n(\gamma; a) \\ &= \frac{1}{2\pi}\left|\int_\gamma \frac{a-b}{(z-a)(z-b)}\, dz\right| \\ &\le \frac{|a-b|}{2\pi}\int_\gamma \frac{|dz|}{|z-a||z-b|}. \end{aligned}$$

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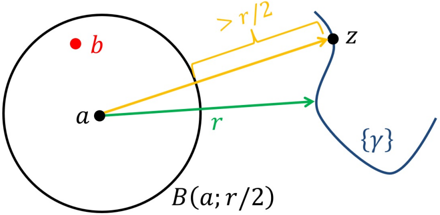

Therefore

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi} \int_\gamma \frac{|dz|}{|z-a||z-b|} < \frac{\delta}{2\pi} \frac{2}{r} \frac{2}{r} V(\gamma) = \frac{2\delta}{\pi r^2} V(\gamma)$$

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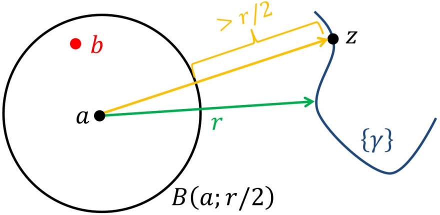

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Now let U be the unbounded component of G. Then there is $R > 0$ such that $U \supset \{z \mid |z| > R\}$. If $\varepsilon > 0$ choose a with $|a| > R$ and $|z - a| > \dfrac{V(\gamma)}{2\pi\varepsilon}$ for all $z \in \{\gamma\}$. Then $|n(\gamma; a)| = \left| \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{1}{z - a}\, dz \right|$ $\leq \dfrac{1}{2\pi} V(\gamma) \left(\dfrac{2\pi\varepsilon}{V(\gamma)} \right) = \varepsilon$. Since ε is arbitrary and $n(\gamma; a)$ is constant for all $a \in G$, then $n(\gamma; a) = 0$ for all $a \in G$. □

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