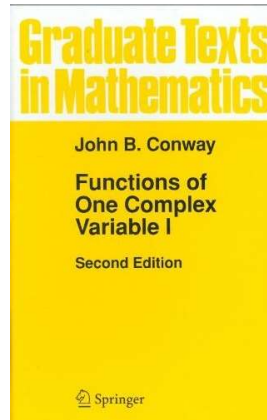


Complex Analysis

Chapter IV. Complex Integration

IV.5. Cauchy's Theorem and Integral Formula—Proofs of Theorems



Lemma IV.5.1

Lemma IV.5.1. Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_{\gamma} \varphi(w)(w-z)^{-m} dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F'_m(z) = mF_{m+1}(z)$.

Proof. Fix $a \notin \{\gamma\}$ and let $r = d(a, \{\gamma\})$. If $b \in \mathbb{C}$ satisfies $|a-b| < \delta < r/2$, then

$$\begin{aligned} F_m(a) - F_m(b) &= \int_{\gamma} \frac{\varphi(w)}{(w-a)^m} dw - \int_{\gamma} \frac{\varphi(w)}{(w-b)^m} dw \\ &= \int_{\gamma} \varphi(w) \left[\frac{1}{(w-a)^m} - \frac{1}{(w-b)^m} \right] dw \\ &= \int_{\gamma} \varphi(w) \left(\frac{1}{w-a} - \frac{1}{w-b} \right) \left[\sum_{k=1}^m \frac{1}{(w-a)^{k-1}} \frac{1}{(w-b)^{m-k}} \right] dw \\ &\quad \text{by algebra} \end{aligned}$$

Lemma IV.5.1 (continued 1)

Proof (continued).

$$\begin{aligned} &= \int_{\gamma} \varphi(w) \frac{(a-b)}{(w-a)(w-b)} \left[\sum_{k=1}^m \frac{1}{(w-a)^{k-1}(w-b)^{m-k}} \right] dw \\ &= \int_{\gamma} \varphi(w)(a-b) \left[\sum_{k=1}^m \frac{1}{(w-a)^k(w-b)^{m-k+1}} \right] dw \\ &= \int_{\gamma} \varphi(w)(a-b) \left[\frac{1}{(w-a)(w-b)^m} + \frac{1}{(w-a)^2(w-b)^{m-1}} \right. \\ &\quad \left. + \cdots + \frac{1}{(w-a)^m(w-b)} \right] dw. \quad (5.2) \end{aligned}$$

(We now mimic the proof of Theorem IV.4.4.) But for $|a-b| < r/2$ and $w \in \{\gamma\}$ we have that $|w-a| \geq r > r/2$ and $|w-b| \geq r > r/2$.

Lemma IV.5.1 (continued 2)

Proof (continued). It follows that

$$\begin{aligned} |F_m(a) - F_m(b)| &\leq |a-b| \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma) \\ &< \delta \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma). \end{aligned}$$

So if $\varepsilon > 0$ is given, then by choosing $\delta > 0$ to be smaller than $r/2$ and $\frac{\varepsilon}{\max_{w \in \{\gamma\}} |\varphi(w)| m V(\gamma)}$, we see that F_m is continuous.

Fix $a \in \mathbb{C} \setminus \{\gamma\} = G$ and $z \in G$, $z \neq a$. From (5.2) (with $b = z$) we have

$$\begin{aligned} \frac{F_m(a) - F_m(z)}{a-z} &= \int_{\gamma} \frac{\varphi(w)}{(w-a)(w-z)^m} dw + \int_{\gamma} \frac{\varphi(w)}{(w-a)^2(w-z)^{m-1}} \\ &\quad + \cdots + \int_{\gamma} \frac{\varphi(w)}{(w-a)^m(w-z)} dw. \end{aligned}$$

Lemma IV.5.1 (continued 3)

Lemma IV.5.1. Let γ be a rectifiable curve and suppose ϕ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_{\gamma} \phi(w)(w - z)^{-m} dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F'_m(z) = mF_{m+1}(z)$.

Proof (continued). By the first part of the proof, each integral on the right hand side is a continuous function of z (z has replaced b in the new notation; to apply the continuity from above, we can let $\varphi(w)$ absorb the power of $w - a$ so that each integral is in the form addressed above) for $z \in G = \mathbb{C} \setminus \{\gamma\}$. So with $z \rightarrow a$ we have

$$F'_m(a) = m \int_{\gamma} \frac{\varphi(w)}{(w - a)^{m+1}} dw = mF_{m+1}(a).$$

Since $a \notin \{\gamma\}$ is arbitrary, the result follows. \square

Theorem IV.5.4

Theorem IV.5.4. Cauchy's Integral Formula (First Version).

Let G be an open subset of the plane and $f : G \rightarrow \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz.$$

Proof. Define $\varphi : G \times G \rightarrow \mathbb{C}$ by $\varphi(z, w) = \frac{f(z) - f(w)}{z - w}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Then φ is continuous and for each $w \in G$, $z \mapsto \varphi(z, w)$ is analytic (by Exercise IV.5.1). Let $H = \{w \in \mathbb{C} \mid n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is continuous and integer-valued on components of $G \setminus \{\gamma\}$ (by Theorem IV.4.4), H is open. Moreover, $H \cup G = \mathbb{C}$ since $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$.

Theorem IV.5.4 (continued 1)

Proof (continued). Define $g : \mathbb{C} \rightarrow \mathbb{C}$ as $g(z) = \int_{\gamma} \varphi(z, w) dw$ if $z \in G$ and $g(z) = \int_{\gamma} \frac{f(w)}{w - z} dw$ if $z \in H$. We need to make sure this piecewise definition is consistent for $z \in G \cap H$. If $z \in G \cap H$ then

$$\begin{aligned} \int_{\gamma} \varphi(z, w) dw &= \int_{\gamma} \frac{f(w) - f(z)}{w - z} dw \\ &= \int_{\gamma} \frac{f(w)}{w - z} dw - f(z) \int_{\gamma} \frac{1}{w - z} dw \\ &= \int_{\gamma} \frac{f(w)}{w - z} dw - f(z)n(\gamma; z) \times 2\pi i \\ &= \int_{\gamma} \frac{f(w)}{w - z} dw \text{ since } n(\gamma; z) = 0 \text{ and } z \in H. \end{aligned}$$

Hence, G is well-defined.

Theorem IV.5.4 (continued 2)

Proof (continued). For $z \in G$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(z) - f(w)$. For $z \in H$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(w)$. So g is an entire function. By Theorem IV.4.4, H contains a neighborhood of ∞ in \mathbb{C}_{∞} . Since f is bounded on $\{\gamma\}$ and $\lim_{z \rightarrow \infty} 1/(w - z) = 0$ uniformly for $w \in \{\gamma\}$ (both follow since $\{\gamma\}$ is compact), we have

$$\begin{aligned} \lim_{z \rightarrow \infty} g(z) &= \lim_{z \rightarrow \infty} \int_{\gamma} \frac{f(w)}{w - z} dw \text{ since for } z \text{ sufficiently large, } z \in H \\ &= \int_{\gamma} \left(\lim_{z \rightarrow \infty} \frac{f(w)}{w - z} \right) dw \text{ by the uniform convergence} \\ &= \int_{\gamma} f(w) \lim_{z \rightarrow \infty} \frac{1}{w - z} dw \\ &= 0 \text{ since } f(w) \text{ is bounded on } \gamma. \end{aligned}$$

Theorem IV.5.4 (continued 3)

Proof (continued). So there exists $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$ (i.e., $z \in \mathbb{C} \setminus \overline{B(0; R)}$). However, g is bounded on $\overline{B(0; R)}$ (since g is continuous and $\overline{B(0; R)}$ is compact). But then, g is a bounded entire function. So by Liouville's Theorem, g is constant. In fact, $g \equiv 0$ since $\lim_{z \rightarrow \infty} g(z) = 0$. So for $a \in G \setminus \{\gamma\}$,

$$\begin{aligned} 0 &= g(a) = \int_{\gamma} \frac{f(z) - f(a)}{z - a} dz \text{ since } a \in G \text{ (} w \text{ replaced with } a\text{)} \\ &= \int_{\gamma} \frac{f(z)}{z - a} dz - f(a) \int_{\gamma} \frac{1}{z - a} dz \\ &= \int_{\gamma} \frac{f(z)}{z - a} dz - f(a)n(\gamma; a)2\pi i. \end{aligned}$$

So,

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz.$$

□

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Theorem IV.5.8

Theorem IV.5.8. Let G be an open set and $f : G \rightarrow \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \dots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \dots + n(\gamma_m; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$ and $k \geq 1$,

$$f^{(k)}(a) \sum_{j=1}^m n(\gamma_j; a) = k! \sum_{j=1}^m \left(\frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z - a)^{k+1}} dz \right).$$

Proof. Differentiate k times the conclusion of Theorem IV.5.6 with respect to a :

$$\frac{d^k}{da^k} \left[f(a) \sum_{j=1}^m n(\gamma_j; a) \right] = \frac{d^k}{da^k} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{k+1}} dz \right].$$

Since $\sum_{j=1}^m n(\gamma_j; a)$ is constant and by repeated application of IV.5.1, the claim follows. □

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Exercise IV.5.5

Exercise IV.5.5. Let γ be a closed rectifiable curve in \mathbb{C} and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_{\gamma} (z - a)^{-n} dz = 0$.

Solution. Define $f(z) \equiv 1$ and $k = n - 1$. Applying Theorem IV.5.8 (with $m = 1$) gives

$$f^{(n-1)}(a)n(\gamma; a) = (n - 1)! \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^n} dz,$$

or $0 = \int_{\gamma} \frac{f(z)}{(z - a)^n} dz$ (since $f^{(n-1)}(a) = 0$). □

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Example

Example. Compute $\int_{|z|=1} e^z z^{-n} dz$. (This is from page 123 of Lars Ahlfors *Complex Analysis*).

Solution. Here, we take $f(z) = e^z$, $a = 0$, $k = n - 1$, and $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then by Corollary IV.5.9,

$$f^{(k)}(a)n(\gamma; a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{k+1}} dz \text{ implies}$$

$$f^{(n-1)}(0)n(\gamma; 0) = \frac{(n - 1)!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - 0)^n} dz \text{ or}$$

$$(e^0)(1) = \frac{(n - 1)!}{2\pi i} \int_{\gamma} \frac{e^z}{z^n} dz. \text{ So } \int_{\gamma} e^z z^{-n} dz = \frac{2\pi i}{(n - 1)!}. \quad \square$$

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Theorem IV.5.10

Theorem IV.5.10. Morera's Theorem.

Let G be a region and let $f : G \rightarrow \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path T in G (i.e., T is a closed polygon with 3 sides); then f is analytic in G .

Proof. Without loss of generality, we assume $G = B(a; R)$ (otherwise, we can write G as a union of disks). We show that f has a primitive F and then we know F is analytic and hence so is $F' = f$. For $z \in G$, define $F(z) = \int_{[a,z]} f(z) dz$. Fix $z_0 \in G$. Then for any $z \in G$, by hypothesis (since a , z , and z_0 form a triangle in G),

$$F(z) = \int_{[a,z]} f(z) dz = \int_{[a,z_0]} f(z) dz + \int_{[z_0,z]} f(z) dz.$$

Hence,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f(z) dz.$$

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Theorem IV.5.10 (continued 1)

Proof (continued). This gives

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0} \int_{[z_0,z]} (f(z) - f(z_0)) dz \\ &= \frac{1}{z - z_0} \int_{[z_0,z]} (f(w) - f(z_0)) dw. \end{aligned}$$

So

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \int_{[z_0,z]} (f(w) - f(z_0)) dw \right| \\ &\leq \frac{1}{|z - z_0|} \int_{[z_0,z]} |f(w) - f(z_0)| |dw| \\ &\leq \frac{|z - z_0|}{|z - z_0|} \sup\{|f(z) - f(z_0)| \mid w \in [z, z_0]\} \\ &= \sup\{|f(w) - f(z_0)| \mid w \in [z, z_0]\}. \end{aligned}$$

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Theorem IV.5.10 (continued 2)

Theorem IV.5.10. Morera's Theorem.

Let G be a region and let $f : G \rightarrow \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path T in G (i.e., T is a closed polygon with 3 sides); then f is analytic in G .

Proof (continued). Since f is continuous,

$$\lim_{z \rightarrow z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

So F is analytic and hence $f = F'$ is analytic. \square

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