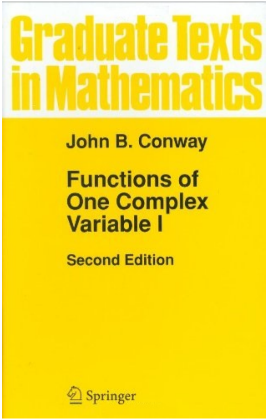
Complex Analysis

Chapter IV. Complex Integration

IV.5. Cauchy's Theorem and Integral Formula—Proofs of Theorems



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Lemma IV.5.1

Lemma IV.5.1. Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\,dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Proof. Fix $a \notin \{\gamma\}$ and let $r = d(a, \{\gamma\})$. If $b \in \mathbb{C}$ satisfies $|a-b| < \delta < r/2$, then

$$F_m(a) - F_m(b) = \int_\gamma \frac{\varphi(w)}{(w-a)^m}\,dw - \int_\gamma \frac{\varphi(w)}{(w-b)^m}\,dw$$

$$= \int_\gamma \varphi(w)\left[\frac{1}{(w-a)^m} - \frac{1}{(w-b)^m}\right] dw$$

$$= \int_\gamma \varphi(w)\left(\frac{1}{w-a} - \frac{1}{w-b}\right)\left[\sum_{k=1}^{m} \frac{1}{(w-a)^{k-1}}\frac{1}{(w-b)^{m-k}}\right] dw$$

by algebra

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Lemma IV.5.1 (continued 1)

Proof (continued).

$$\begin{aligned}
&= \int_\gamma \varphi(w)\frac{(a-b)}{(w-a)(w-b)}\left[\sum_{k=1}^{m}\frac{1}{(w-a)^{k-1}(w-b)^{m-k}}\right] dw \\
&= \int_\gamma \varphi(w)(a-b)\left[\sum_{k=1}^{m}\frac{1}{(w-a)^{k}(w-b)^{m-k+1}}\right] dw \\
&= \int_\gamma \varphi(w)(a-b)\left[\frac{1}{(w-a)(w-b)^m}+\frac{1}{(w-a)^2(w-b)^{m-1}}\right. \\
&\qquad\qquad \left. +\cdots+\frac{1}{(w-a)^m(w-b)}\right] dw. \qquad (5.2)
\end{aligned}$$

(We now mimic the proof of Theorem IV.4.4.) But for $|a-b|<r/2$ and $w\in\{\gamma\}$ we have that $|w-a|\geq r>r/2$ and $|w-b|\geq r>r/2$.

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$$\begin{aligned}
&= \int_\gamma \varphi(w)\frac{(a-b)}{(w-a)(w-b)}\left[\sum_{k=1}^{m}\frac{1}{(w-a)^{k-1}(w-b)^{m-k}}\right]dw \\
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(We now mimic the proof of Theorem IV.4.4.) But for $|a-b|<r/2$ and $w\in\{\gamma\}$ we have that $|w-a|\geq r>r/2$ and $|w-b|\geq r>r/2$.

Lemma IV.5.1 (continued 2)

Proof (continued). It follows that

$$\begin{aligned} |F_m(a) - F_m(b)| &\le |a-b| \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma) \\ &< \delta \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma). \end{aligned}$$

So if $\varepsilon > 0$ is given, then by choosing $\delta > 0$ to be smaller than $r/2$ and $\dfrac{(r/2)^{m+1}\varepsilon}{\max_{w \in \{\gamma\}} |\varphi(w)| m V(\gamma)}$, we see that F_m is continuous.

Fix $a \in \mathbb{C} \setminus \{\gamma\} = G$ and $z \in G$, $z \ne a$. From (5.2) (with $b = z$) we have

$$\begin{aligned} \frac{F_m(a) - F_m(z)}{a - z} &= \int_\gamma \frac{\varphi(w)}{(w-a)(w-z)^m}\, dw + \int_\gamma \frac{\varphi(w)}{(w-a)^2(w-z)^{m-1}} \\ &\quad + \cdots + \int_\gamma \frac{\varphi(w)}{(w-a)^m(w-z)}\, dw. \end{aligned}$$

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Fix $a \in \mathbb{C} \setminus \{\gamma\} = G$ and $z \in G$, $z \ne a$. From (5.2) (with $b = z$) we have

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Lemma IV.5.1 (continued 3)

Lemma IV.5.1. Let γ be a rectifiable curve and suppose ϕ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_\gamma \phi(w)(w-z)^{-m}\, dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Proof (continued). By the first part of the proof, each integral on the right hand side is a continuous function of z (z has replaced b in the new notation; to apply the continuity from above, we can let $\varphi(w)$ absorb the power of $w - a$ so that each integral is in the form addressed above) for $z \in G = \mathbb{C} \setminus \{\gamma\}$. So with $z \to a$ we have

$$F_m'(a) = m \int_\gamma \frac{\varphi(w)}{(w-a)^{m+1}}\, dw = mF_{m+1}(a).$$

Since $a \notin \{\gamma\}$ is arbitrary, the result follows. $\square$

Theorem IV.5.4

Theorem IV.5.4. Cauchy's Integral Formula (First Version).
Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Proof. Define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z, w) = \frac{f(z)-f(w)}{z-w}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Then φ is continuous and for each $w \in G$, $z \to \varphi(z, w)$ is analytic (by Exercise IV.5.1).

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Theorem IV.5.4 (continued 1)

Proof (continued). Define $g : \mathbb{C} \to \mathbb{C}$ as $g(z) = \int_\gamma \varphi(z,w)\,dw$ if $z \in G$ and $g(z) = \int_\gamma \frac{f(w)}{w-z}\,dw$ if $z \in H$. We need to make sure this piecewise definition is consistent for $z \in G \cap H$. If $z \in G \cap H$ then

$$\begin{aligned}
\int_\gamma \varphi(z,w)\,dw &= \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw \\
&= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)\int_\gamma \frac{1}{w-z}\,dw \\
&= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)n(\gamma;z)\times 2\pi i \\
&= \int_\gamma \frac{f(w)}{w-z}\,dw \text{ since } n(\gamma;z)=0 \text{ and } z \in H.
\end{aligned}$$

Hence, G is well-defined.

Theorem IV.5.4 (continued 1)

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Theorem IV.5.4 (continued 2)

Proof (continued). For $z \in G$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(z) - f(w)$. For $z \in H$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(w)$. So g is an entire function. By Theorem IV.4.4, H contains a neighborhood of ∞ in $\mathbb{C}_\infty$. Since f is bounded on $\{\gamma\}$ and $\lim_{z\to\infty} 1/(w-z) = 0$ uniformly for $w \in \{\gamma\}$ (both follow since $\{\gamma\}$ is compact), we have

$$\begin{aligned}
\lim_{z\to\infty} g(z) &= \lim_{z\to\infty} \int_\gamma \frac{f(w)}{w-z}\,dw \text{ since for } z \text{ sufficiently large, } z \in H \\
&= \int_\gamma \left(\lim_{z\to\infty} \frac{f(w)}{w-z} \right) dw \text{ by the uniform convergence} \\
&= \int_\gamma f(w) \lim_{z\to\infty} \frac{1}{w-z}\,dw \\
&= 0 \text{ since } f(w) \text{ is bounded on } \gamma.
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Proof (continued). So there exists $R > 0$ such that $|g(z)| \le 1$ for $|z| \ge R$ (i.e., $z \in \mathbb{C} \setminus B(0; R)$). However, g is bounded on $\overline{B}(0; R)$ (since g is continuous and $\overline{B}(0; R)$ is compact). But then, g is a bounded entire function. So by Liouville's Theorem, g is constant. In fact, $g \equiv 0$ since $\lim_{z\to\infty} g(z) = 0$. So for $a \in G \setminus \{\gamma\}$,

$$\begin{aligned} 0 &= g(a) = \int_\gamma \frac{f(z) - f(a)}{z - a}\, dz \text{ since } a \in G \ (w \text{ replaced with } a) \\ &= \int_\gamma \frac{f(z)}{z-a}\, dz - f(a) \int_\gamma \frac{1}{z-a}\, dz \\ &= \int_\gamma \frac{f(z)}{z-a}\, dz - f(a) n(\gamma; a) 2\pi i. \end{aligned}$$

So,

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

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So,

$$n(\gamma;a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz.$$

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Theorem IV.5.8

Theorem IV.5.8. Let G be an open set and $f : G \to \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$ and $k \geq 1$,

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \left(\frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\, dz \right).$$

Proof. Differentiate k times the conclusion of Theorem IV.5.6 with respect to a:

$$\frac{d^k}{da^k}\left[f(a) \sum_{j=1}^{m} n(\gamma_j; a) \right] = \frac{d^k}{da^k}\left[\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)}\, dz \right].$$

Since $\sum_{j=1}^{m} n(\gamma_j; a)$ is constant and by repeated application of IV.5.1, the claim follows. □

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Exercise IV.5.5

Exercise IV.5.5. Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_\gamma (z-a)^{-n}\, dz = 0$.

Solution. Define $f(z) \equiv 1$ and $k = n-1$. Applying Theorem IV.5.8 (with $m = 1$) gives

$$f^{(n-1)}(a)n(\gamma; a) = (n-1)! \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^n}\, dz,$$

or $0 = \displaystyle\int_\gamma \frac{f(z)}{(z-a)^n}\, dz$ (since $f^{(n-1)}(a) = 0$). □

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Example

Example. Compute $\int_{|z|=1} e^z z^{-n}\, dz$. (This is from page 123 of Lars Ahlfors *Complex Analysis*).

Solution. Here, we take $f(z) = e^z$, $a = 0$, $k = n-1$, and $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then by Corollary IV.5.9,

$f^{(k)}(a)n(\gamma; a) = \dfrac{k!}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{(z-a)^{k+1}}\, dz$ implies

$f^{(n-1)}(0)n(\gamma; 0) = \dfrac{(n-1)!}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{(z-0)^{n}}\, dz$ or

$(e^0)(1) = \dfrac{(n-1)!}{2\pi i}\displaystyle\int_\gamma \dfrac{e^z}{z^n}\, dz$. So $\displaystyle\int_\gamma e^z z^{-n}\, dz = \dfrac{2\pi i}{(n-1)!}$. □

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Theorem IV.5.10

Theorem IV.5.10. Morera's Theorem.
Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path T in G (i.e., T is a closed polygon with 3 sides); then f is analytic in G.

Proof. Without loss of generality, we assume $G = B(a; R)$ (otherwise, we can write G as a union of disks). We show that f has a primitive F and then we know F is analytic and hence so is $F' = f$. For $z \in G$, define $F(z) = \int_{[a,z]} f(z)\,dz$. Fix $z_0 \in G$. Then for any $z \in G$, by hypothesis (since a, z, and z_0 form a triangle in G),

$$F(z) = \int_{[a,z]} f(z)\,dz = \int_{[a,z_0]} f(z)\,dz + \int_{[z_0,z]} f(z)\,dz.$$

Hence,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f(z)\,dz.$$

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Hence,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f(z)\,dz.$$

Theorem IV.5.10 (continued 1)

Proof (continued). This gives

$$\begin{aligned} \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} (f(z)-f(z_0))\,dz \\ &= \frac{1}{z-z_0}\int_{[z_0,z]} (f(w)-f(z_0))\,dw. \end{aligned}$$

So

$$\begin{aligned} \left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &= \left|\frac{1}{z-z_0}\int_{[z_0,z]} (f(w)-f(z_0))\,dw\right| \\ &\le \frac{1}{|z-z_0|}\int_{[z_0,z]} |f(w)-f(z_0)|\,|dw| \\ &\le \frac{|z-z_0|}{|z-z_0|}\sup\{|f(z)-f(z_0)| \mid w \in [z,z_0]\} \\ &= \sup\{|f(w)-f(z_0)| \mid w \in [z,z_0]\}. \end{aligned}$$

Theorem IV.5.10 (continued 2)

Theorem IV.5.10. Morera's Theorem.
Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path T in G (i.e., T is a closed polygon with 3 sides); then f is analytic in G.

Proof (continued). Since f is continuous,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

So F is analytic and hence $f = F'$ is analytic. □