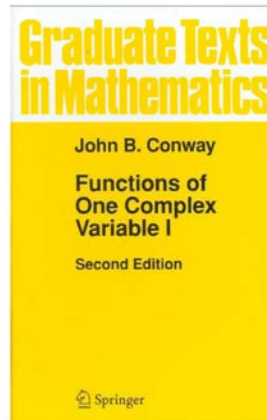


Complex Analysis

Chapter IV. Complex Integration

IV.6. The Homotopic Version of Cauchy's Theorem and Simple Connectivity—Proofs of Theorems



Proposition IV.6.4

Proposition IV.6.4. Let G be an open set which is a -star shaped. If γ_0 is the curve which is constantly equal to a (that is, $\gamma_0(t) = a$ for $t \in [0, 1]$), then every closed rectifiable curve in G is homotopic to γ_0 .

Proof. Let γ_1 be a closed rectifiable curve in G and put $\Gamma(s, t) = t\gamma_1(s) + (1 - t)a$. So for each fixed s , $\Gamma(s, t)$ is the segment $[\gamma_1(s), a]$. Since G is a -star shaped, $\Gamma(s, t) \in G$ for all $s, t \in [0, 1]$. Γ satisfies the required properties of a homotopy “clearly.” \square

Theorem IV.6.7

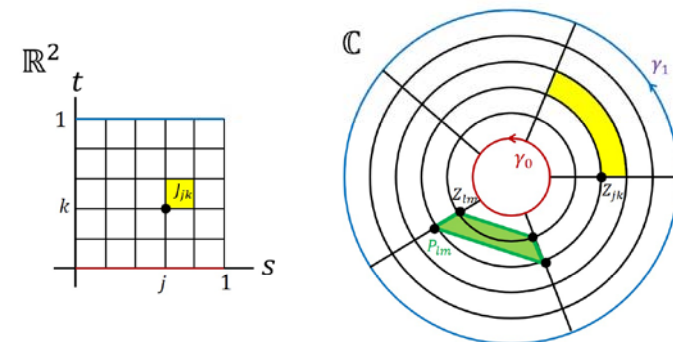
Theorem IV.6.7. Cauchy's Theorem (Third Version).

If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic on G .

Proof. Let $\gamma : I^2 \rightarrow G$ (where $I = [0, 1]$) be the homotopy function from γ_0 to γ_1 . Since Γ is continuous and I^2 is compact, Γ is uniformly continuous and $\Gamma(I^2)$ is a compact subset of G . Since $\mathbb{C} \setminus G$ is closed, the distance from $\Gamma(I^2)$ to $\mathbb{C} \setminus G$ is positive (by Theorem II.5.17), so $d(\Gamma(I^2), \mathbb{C} \setminus G) = r > 0$. So by the uniform continuity of Γ , there is $n \in \mathbb{N}$ such that if $d((s, t), (s', t')) < 2/n$ (or $(s - s')^2 + (t - t')^2 < 4/n^2$) then $|\Gamma(s, t) - \Gamma(s', t')| < r$. Define $Z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right)$ for $0 \leq j \leq n, 0 \leq k \leq n$ and put $J_{jk} = \left[\frac{j}{n}, \frac{j+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$ for $0 \leq j \leq n-1, 0 \leq k \leq n-1$.

Theorem IV.6.7 (continued 1)

Proof (continued).



Since the diameter of square J_{jk} is $\sqrt{2}/n$, then all points in J_{jk} are within $\sqrt{2}/n < 2/n$ and hence all the images of points in J_{jk} under Γ are within r of each other; that is, $\Gamma(J_{jk}) \subset B(Z_{jk}; r)$. Let P_{jk} be the closed quadrilateral $[Z_{j,k}, Z_{j+1,k}, Z_{j+1,k+1}, Z_{j,k+1}, Z_{j,k}]$. Because disks are convex, $P_{jk} \subset B(Z_{jk}; r)$.

Theorem IV.6.7 (continued 2)

Proof (continued). By Proposition IV.2.15,

$$\int_{P_{jk}} f(z) dz = 0 \quad (6.8)$$

for any function f analytic in G . Let Q_k be the closed polygon $[Z_{0k}, Z_{1k}, \dots, Z_{nk}]$ (a polygon approximation of closed path $\Gamma(s, k/n)$ for $s \in [0, 1]$). We will show that

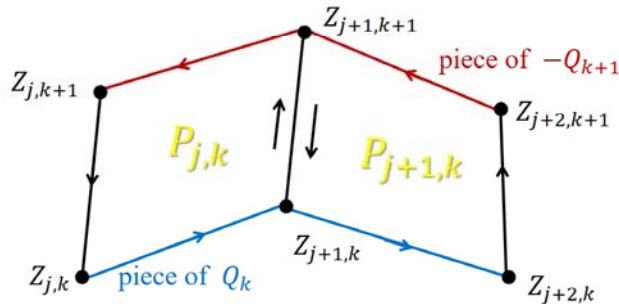
$$\int_{\gamma_0} f(z) dz = \int_{Q_0} f(z) dz = \int_{Q_1} f(z) dz = \dots = \int_{Q_n} f(z) dz = \int_{\gamma_1} f(z) dz.$$

Define $\sigma_j(t) \in \gamma_0(t)$ for $t \in [j/n, (j+1)/n]$ (so this is just a partition of γ_0).

Theorem IV.6.7 (continued 4)

Proof (continued). By (6.8): $\sum_{j=0}^{n-1} \left(\int_{P_{jk}} f(z) dz \right) = 0. \quad (6.9)$

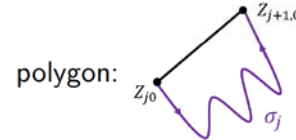
For j and $j+1$ we have:



Notice that the right hand part of P_{jk} and the left hand part of $P_{j+1,k}$ lead to integrals which cancel each other out.

Theorem IV.6.7 (continued 3)

Proof (continued). Then $\sigma_j + [Z_{j+1,0}, Z_{j0}]$, which is σ_j followed by the



is a closed rectifiable curve in the disk

$B(Z_{j0}; r) \subset G$ (again, uniform continuity of Γ). So

$$\int_{\sigma_j + [Z_{j+1,0}, Z_{j0}]} f(z) dz = 0, \text{ or}$$

$$\int_{\sigma_j} f(z) dz = - \int_{[Z_{j+1,0}, Z_{j0}]} f(z) dz = \int_{[Z_{j0}, Z_{j+1,0}]} f(z) dz.$$

Summing both sides for $j = 0, 1, \dots, n-1$ yields

$$\int_{\gamma_0} f(z) dz = \int_{Q_0} f(z) dz. \text{ Similarly, } \int_{\gamma_1} f(z) dz = \int_{Q_n} f(z) dz.$$

Theorem IV.6.7 (continued 5)

Theorem IV.6.7. Cauchy's Theorem (Third Version).

If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic on G .

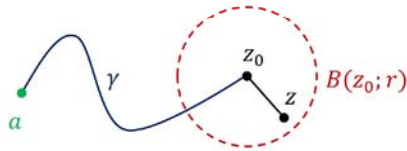
Proof (continued). Also, $Z_{0k} = \Gamma(0, k/n) = \Gamma(1, k/n) = Z_{nk}$ (by the definition of the homotopy) so that $[Z_{0,k+1}, Z_{0k}] = -[Z_{1k}, Z_{1,k+1}]$. So by (6.9), we see that $\int_{Q_k} f(z) dz = \int_{Q_{k+1}} f(z) dz$. Therefore

$$\int_{\gamma_0} f(z) dz = \int_{Q_0} f(z) dz = \int_{Q_1} f(z) dz = \dots = \int_{Q_n} f(z) dz = \int_{\gamma_1} f(z) dz. \quad \square$$

Corollary IV.6.16

Corollary IV.6.16. If open G is simply connected and $f : G \rightarrow \mathbb{C}$ is analytic in G then f has a primitive in G .

Proof. Fix point $a \in G$ and let γ_1 and γ_2 be any two rectifiable curves in G from a to (variable) point $z \in G$. Since G is open and connected, there is always a path between any two points in G by Theorem II.2.3; this is called "path connected"). Then by the Cauchy Theorem (Fourth Version), $0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$, and so $\int_{\gamma_1} f = \int_{\gamma_2} f$. So if we define $F(z) = \int_{\gamma} f$ where γ is any path from a to z , then F is well-defined (i.e., independent of γ). Of course, F is a primitive of f . If $z_0 \in G$ and $r > 0$ is such that $B(z_0; r) \subset G$ (remember, G is open), and then let γ be a path from a to z_0 . For $z \in B(z_0; r)$, let $\gamma_z = \gamma + [z_0, z]$:



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Corollary IV.6.16 (continued 1)

Proof (continued). Then

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{z - z_0} \left(\int_{\gamma_z} f(z) dz - \int_{\gamma} f(z) dz \right) \\ &= \frac{1}{z - z_0} \int_{\gamma_z - \gamma} f(z) dz = \frac{1}{z - z_0} \int_{[z_0, z]} f(z) dz. \end{aligned}$$

Mimicing the proof of Morera's Theorem, this gives

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) &= \frac{1}{z - z_0} \int_{[z_0, z]} (f(z) - f(z_0)) dz \\ &= \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) dw. \end{aligned}$$

So

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = \left| \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) dw \right|$$

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Corollary IV.6.16 (continued 2)

Corollary IV.6.16. If open G is simply connected and $f : G \rightarrow \mathbb{C}$ is analytic in G then f has a primitive in G .

Proof (continued).

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) dw \right| \\ &\leq \frac{1}{|z - z_0|} \int_{[z_0, z]} |f(w) - f(z_0)| |dw| \\ &\leq \frac{1}{|z - z_0|} |z - z_0| \sup\{|f(w) - f(z_0)| \mid w \in [z, z_0]\} \\ &= \sup\{|f(w) - f(z_0)| \mid w \in [z, z_0]\}. \end{aligned}$$

Since f is continuous, $\lim_{z \rightarrow z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0)$. That is, $F' = f$ and f has a primitive. \square

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Corollary IV.6.17

Corollary IV.6.17. Let G be simply connected and let $f : G \rightarrow \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) = \exp(g(z))$ (i.e., g is a branch of $\log(f(z))$ on G). If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof. Since $f(z) \neq 0$ for $z \in G$, then f'/f is analytic on G . So by Corollary 6.16, f'/f has a primitive g_1 . If $h(z) = \exp(g_1(z))$ then h is analytic and never 0. So f/h is analytic with derivative

$$\frac{f'(z)h(z) - f(z)h'(z)}{(h(z))^2}.$$

Next, $h'(z) = \exp(g_1(z))[g_1'(z)] = g_1'(z)h(z)$ and so

$$\begin{aligned} f'(z)h(z) - f(z)h'(z) &= f'(z)h(z) - f(z)g_1'(z)h(z) \\ &= f'(z)h(z) - f(z)(f'(z)/f(z))h(z) \equiv 0. \end{aligned}$$

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Corollary IV.6.17 (continued)

Corollary IV.6.17. Let G be simply connected and let $f : G \rightarrow \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \rightarrow \mathbb{C}$ such that $f(z) = \exp(g(z))$ (i.e., g is a branch of $\log(f(z))$ on G). If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof (continued). So $\frac{d}{dz} \left[\frac{f(z)}{h(z)} \right] = 0$ and $\frac{f(z)}{h(z)}$ is constant on G . That is, $f(z)/h(z) = c$ or $f(z) = ch(z) = c \exp(g_1(z))$ for some constant c and for c' where $c = \exp(c')$, $f(z) = \exp(g_1(z) + c')$. So $g(z) = g_1(z) + c'$ is a branch of $\log f(z)$ on G . If $z_0 \in G$ and $e^{w_0} = f(z_0)$ then $w_0 = \log^*(f(z_0))$ for some branch of $\log(f(z))$ on G . Since $g(z)$ is a branch of $\log(f(z))$ on G then by Proposition III.2.19 there is $k \in \mathbb{Z}$ such that $\log^*(f(z)) = g(z) + 2\pi ik$ and we may choose this as g to get $g(z_0) = \log^*(f(z_0)) = w_0$. \square