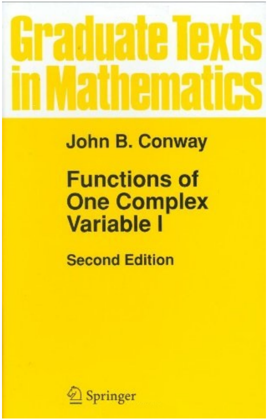
Complex Analysis

Chapter IV. Complex Integration

IV.6. The Homotopic Version of Cauchy's Theorem and Simple Connectivity—Proofs of Theorems



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Proof. Let γ_1 be a closed rectifiable curve in G and put $\Gamma(s, t) = t\gamma_1(s) + (1 - t)a$. So for each fixed s, $\Gamma(s, t)$ is the segment $[\gamma_1(s), a]$. Since G is a-star shaped, $\Gamma(s, t) \in G$ for all $s, t \in [0, 1]$. Γ satisfies the required properties of a homotopy "clearly." □

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Theorem IV.6.7

Theorem IV.6.7. Cauchy's Theorem (Third Version).
If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic on G.

Proof. Let $\gamma : I^2 \to G$ (where $I = [0,1]$) be the homotopy function from γ_0 to γ_1. Since Γ is continuous and I^2 is compact, Γ is uniformly continuous and $\Gamma(I^2)$ is a compact subset of G. Since $\mathbb{C} \setminus G$ is closed, the distance from $\Gamma(I^2)$ to $\mathbb{C} \setminus G$ is positive (by Theorem II.5.17), so $d(\Gamma(I^2), \mathbb{C} \setminus G) = r > 0$.

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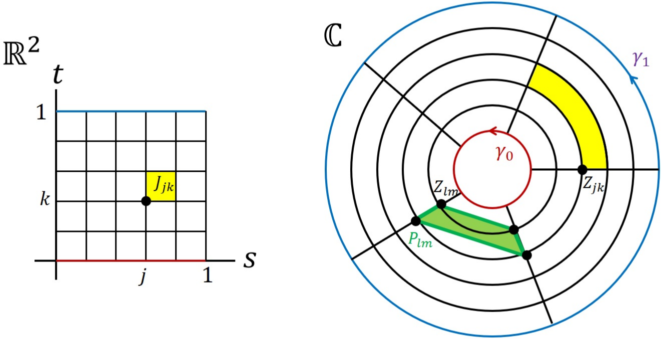
Theorem IV.6.7 (continued 1)

Proof (continued).



Since the diameter of square J_{jk} is $\sqrt{2}/n$, then all points in J_{jk} are within $\sqrt{2}/n < 2/n$ and hence all the images of points in J_{jk} under Γ are within r of each other; that is, $\Gamma(J_{jk}) \subset B(Z_{jk}; r)$. Let P_{jk} be the closed quadrilateral $[Z_{j,k}, Z_{j+1,k}, Z_{j+1,k+1}, Z_{j,k+1}, Z_{j,k}]$. Because disks are convex, $P_{jk} \subset B(Z_{jk}; r)$.

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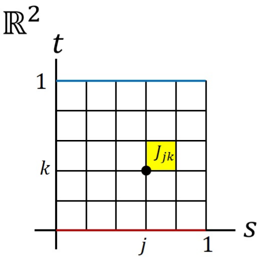

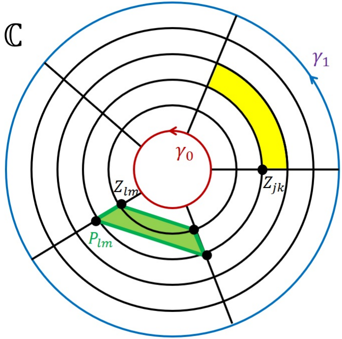

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Theorem IV.6.7 (continued 2)

Proof (continued). By Proposition IV.2.15,

$$\int_{P_{jk}} f(z)\,dz = 0 \qquad (6.8)$$

for any function f analytic in G. Let Q_k be the closed polygon $[Z_{0k}, Z_{1k}, \ldots, Z_{nk}]$ (a polygon approximation of closed path $\Gamma(s, k/n)$ for $s \in [0,1]$). We will show that

$$\int_{\gamma_0} f(z)\,dz = \int_{Q_0} f(z)\,dz = \int_{Q_1} f(z)\,dz = \cdots = \int_{Q_n} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Define $\sigma_j(t) \in \gamma_0(t)$ for $t \in [j/n, (j+1)/n]$ (so this is just a partition of γ_0).

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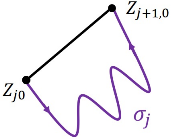
Proof (continued). Then $\sigma_j + [Z_{j+1,0}, Z_{j0}]$, which is σ_j followed by the polygon: 

is a closed rectifiable curve in the disk $B(Z_{j0}; r) \subset G$ (again, uniform continuity of Γ). So $\int_{\sigma_j + [Z_{j+1,0}, Z_{j0}]} f(z)\,dz = 0$, or

$$\int_{\sigma_j} f(z)\,dz = -\int_{[Z_{j+1,0}, Z_{j0}]} f(z)\,dz = \int_{[Z_{j0}, Z_{j+1,0}]} f(z)\,dz.$$

Summing both sides for $j = 0, 1, \ldots, n-1$ yields $\int_{\gamma_0} f(z)\,dz = \int_{Q_0} f(z)\,dz$. Similarly, $\int_{\gamma_1} f(z)\,dz = \int_{Q_n} f(z)\,dz$.

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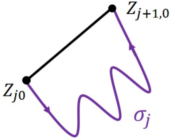
Proof (continued). Then $\sigma_j + [Z_{j+1,0}, Z_{j0}]$, which is σ_j followed by the polygon: 

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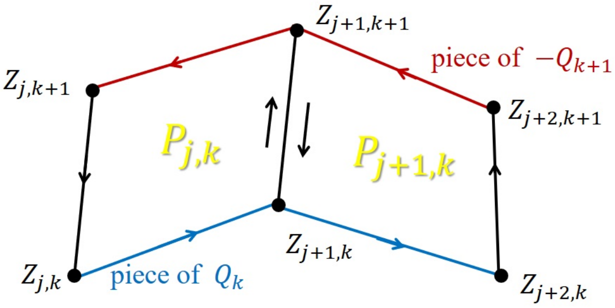
Theorem IV.6.7 (continued 4)

Proof (continued). By (6.8): $\sum_{j=0}^{n-1}\left(\int_{P_{jk}} f(z)\,dz\right) = 0.$ (6.9)

For j and $j+1$ we have:



Notice that the right hand part of P_{jk} and the left hand part of $P_{j+1,k}$ lead to integrals which cancel each other out.

Theorem IV.6.7 (continued 5)

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Corollary IV.6.16

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Proof. Fix point $a \in G$ and let γ_1 and γ_2 be any two rectifiable curves in G from a to (variable) point $z \in G$. Since G is open and connected, there is always a path between any two points in G by Theorem II.2.3; this is called "path connected"). Then by the Cauchy Theorem (Fourth Version), $0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$, and so $\int_{\gamma_1} f = \int_{\gamma_2} f$.

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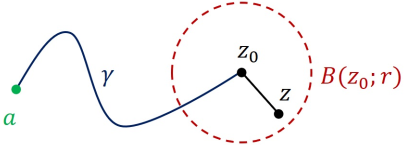

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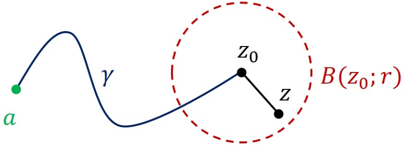

Corollary IV.6.16 (continued 1)

Proof (continued). Then

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\left(\int_{\gamma_z} f(z)\,dz - \int_{\gamma} f(z)\,dz\right)$$
$$= \frac{1}{z-z_0}\int_{\gamma_z-\gamma} f(z)\,dz = \frac{1}{z-z_0}\int_{[z_0,z]} f(z)\,dz.$$

Mimicing the proof of Morera's Theorem, this gives

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} (f(z)-f(z_0))\,dz$$
$$= \frac{1}{z-z_0}\int_{[z_0,z]} (f(w)-f(z_0))\,dw.$$

So

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| = \left|\frac{1}{z-z_0}\int_{[z_0,z]} (f(w)-f(z_0))\,dw\right|$$

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Corollary IV.6.16 (continued 2)

Corollary IV.6.16. If open G is simply connected and $f : G \to \mathbb{C}$ is analytic in G then f has a primitive in G.

Proof (continued).

$$\begin{aligned} \left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &= \left|\frac{1}{z-z_0}\int_{[z_0,z]}(f(w)-f(z_0))\,dw\right| \\ &\le \frac{1}{|z-z_0|}\int_{[z_0,z]}|f(w)-f(z_0)|\,|dw| \\ &\le \frac{1}{|z-z_0|}|z-z_0|\sup\{|f(w)-f(z_0)| \mid w\in[z,z_0]\} \\ &= \sup\{|f(w)-f(z_0)| \mid w\in[z,z_0]\}. \end{aligned}$$

Since f is continuous, $\lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$. That is, $F' = f$ and f has a primitive. □

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Proof (continued).

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0))\, dw \right| \\ &\leq \frac{1}{|z - z_0|} \int_{[z_0, z]} |f(w) - f(z_0)|\, |dw| \\ &\leq \frac{1}{|z - z_0|} |z - z_0| \sup\{|f(w) - f(z_0)| \mid w \in [z, z_0]\} \\ &= \sup\{|f(w) - f(z_0)| \mid w \in [z, z_0]\}. \end{aligned}$$

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Corollary IV.6.17

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Proof. Since $f(z) \neq 0$ for $z \in G$, then f'/f is analytic on G. So by Corollary 6.16, f'/f has a primitive g_1. If $h(z) = \exp(g_1(z))$ then h is analytic and never 0. So f/h is analytic with derivative

$$\frac{f'(z)h(z) - f(z)h'(z)}{(h(z))^2}.$$

Next, $h'(z) = \exp(g_1(z))[g_1'(z)] = g_1'(z)h(z)$ and so

$$f'(z)h(z) - f(z)h'(z) = f'(z)h(z) - f(z)g_1'(z)h(z)$$

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Proof (continued). So $\dfrac{d}{dz}\left[\dfrac{f(z)}{h(z)}\right] = 0$ and $\dfrac{f(z)}{h(z)}$ is constant on G. That is, $f(z)/h(z) = c$ or $f(z) = ch(z) = c\exp(g_1(z))$ for some constant c and for c' where $c = \exp(c')$, $f(z) = \exp(g_1(z) + c')$. So $g(z) = g_1(z) + c'$ is a branch of $\log f(z)$ on G.

Corollary IV.6.17 (continued)

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Proof (continued). So $\dfrac{d}{dz}\left[\dfrac{f(z)}{h(z)}\right] = 0$ and $\dfrac{f(z)}{h(z)}$ is constant on G. That is, $f(z)/h(z) = c$ or $f(z) = ch(z) = c\exp(g_1(z))$ for some constant c and for c' where $c = \exp(c')$, $f(z) = \exp(g_1(z) + c')$. So $g(z) = g_1(z) + c'$ is a branch of $\log f(z)$ on G. If $z_0 \in G$ and $e^{w_0} = f(z_0)$ then $w_0 = \log^*(f(z_0))$ for some branch of $\log(f(z))$ on G. Since $g(z)$ is a branch of $\log(f(z))$ on G then by Proposition III.2.19 there is $k \in \mathbb{Z}$ such that $\log^*(f(z)) = g(z) + 2\pi ik$ and we may choose this as g to get $g(z_0) = \log^*(f(z_0)) = w_0$. □

Corollary IV.6.17 (continued)

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Proof (continued). So $\frac{d}{dz}\left[\frac{f(z)}{h(z)}\right] = 0$ and $\frac{f(z)}{h(z)}$ is constant on G. That is, $f(z)/h(z) = c$ or $f(z) = ch(z) = c\exp(g_1(z))$ for some constant c and for c' where $c = \exp(c')$, $f(z) = \exp(g_1(z) + c')$. So $g(z) = g_1(z) + c'$ is a branch of $\log f(z)$ on G. If $z_0 \in G$ and $e^{w_0} = f(z_0)$ then $w_0 = \log^*(f(z_0))$ for some branch of $\log(f(z))$ on G. Since $g(z)$ is a branch of $\log(f(z))$ on G then by Proposition III.2.19 there is $k \in \mathbb{Z}$ such that $\log^*(f(z)) = g(z) + 2\pi ik$ and we may choose this as g to get $g(z_0) = \log^*(f(z_0)) = w_0$. □