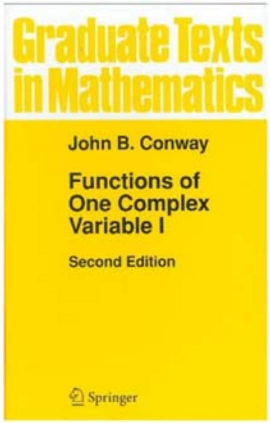
# Complex Analysis

**Chapter IV. Complex Integration**

IV.7. Counting Zeros; The Open Mapping Theorem—Proofs of Theorems








## Theorem IV.7.2

**Theorem IV.7.2.** Let $G$ be a region and let $f$ be an analytic function on $G$ with zeros $a_1, a_2, \ldots, a_m$ (repeated according to multiplicity). If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any point $a_k$ and if $\gamma \sim 0$ in $G$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^m n(\gamma; a_k).$$

**Proof.** By Corollary IV.3.9, we can factor $f$ as $f(z) = (z-a_1)(z-a_2)\cdots(z-a_m)g(z)$ where $g$ is analytic on $G$ and $g(z) \neq 0$ in $G$. We get from the product rule:

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}.$$





## Theorem IV.7.2 (continued)

**Proof.** Since $\gamma \approx 0$,

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= \int_\gamma \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}\right) dz\\ &= 2\pi i(n(\gamma;a_1) + n(\gamma;a_2) + \cdots + n(\gamma;a_m)) + \int_\gamma \frac{g'(z)}{g(z)}\,dz\\ &\qquad \text{by the definition of winding number}\\ &= 2\pi i(n(\gamma;a_1) + n(\gamma;a_2) + \cdots + n(\gamma;a_m)) + 0\\ &\qquad \text{by Corollary IV.1.22}\end{aligned}$$

and the result follows. □





## Example IV.7.A

**Example IV.7.A.** Evaluate $\displaystyle\int_\gamma \frac{2z+1}{z^2+z+1}\,dz$ where $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$.

**Solution.** Well, $z^2+z+1$ has zeros at two points $\omega_1$ and $\omega_2$ where $|\omega_1| = |\omega_2| = 1$ (since $z^3 - 1 = (z-1)(z^2+z+1)$). So by Theorem IV.7.2 with $f(z) = z^2+1$, we have

$$\int_\gamma \frac{2z+1}{z^2+z+1}\,dz = 2\pi i(n(\gamma;\omega_1) + n(\gamma;\omega_2)) = 4\pi i.$$

(Does $\displaystyle\frac{2z+1}{z^2+z+1}$ have a primitive on $\gamma$?) □

## Theorem IV.7.5

**Theorem IV.7.5. The Open Mapping Theorem.**
Let $G$ be a region (open connected set) and suppose that $f$ is a nonconstant analytic function on $G$. Then for any open set $U$ in $G$, $F(U)$ is open.

**Proof.** Let $U \subset G$ be open and let $\alpha \in f(U)$. Then $\alpha = f(a)$ for some $a \in U$. Since $U$ is open, there is $\varepsilon > 0$ with $B(a;\varepsilon) \subset U$ and by Theorem IV.7.4 this $\varepsilon$ can be chosen, along with $\delta > 0$, such that $f(B(a;\varepsilon)) \supset B(\alpha;\delta)$. So $B(\alpha;\delta) \subset f(B(a;\varepsilon)) \subset f(U)$, and therefore $f(U)$ is open. □





## Corollary IV.7.6

**Corollary IV.7.6.** Suppose $f : G \to \mathbb{C}$ is one to one, analytic and $f(G) = \Omega$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(\omega) = 1/f'(z)$ where $\omega = f(z)$.

**Proof.** By the Open Mapping Theorem (Theorem IV.7.5), $f^{-1}$ has inverse images of open sets open and so $f^{-1}$ is continuous, AND $f(G) = \Omega$ is open. Since $z = f^{-1}(f(z))$, the result follows by Proposition III.2.20. □