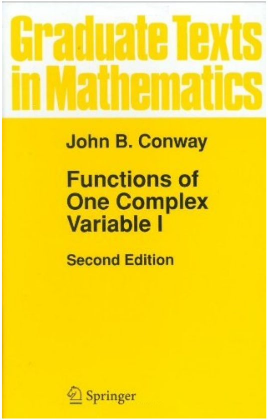
Complex Analysis

Chapter IV. Complex Integration

IV.7. Counting Zeros; The Open Mapping Theorem—Proofs of Theorems



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Theorem IV.7.2

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$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} n(\gamma; a_k).$$

Proof. By Corollary IV.3.9, we can factor f as $f(z) = (z - a_1)(z - a_2) \cdots (z - a_m) g(z)$ where g is analytic on G and $g(z) \neq 0$ in G. We get from the product rule:

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \frac{1}{z - a_2} + \cdots + \frac{1}{z - a_m} + \frac{g'(z)}{g(z)}.$$

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Theorem IV.7.2 (continued)

Proof. Since $\gamma \approx 0$,

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= \int_\gamma \left(\frac{1}{z-a_1}+\frac{1}{z-a_2}+\cdots+\frac{1}{z-a_m}+\frac{g'(z)}{g(z)}\right)\,dz \\ &= 2\pi i(n(\gamma;a_1)+n(\gamma;a_2)+\cdots+n(\gamma;a_m))+\int_\gamma \frac{g'(z)}{g(z)}\,dz \\ &\qquad \text{by the definition of winding number} \\ &= 2\pi i(n(\gamma;a_1)+n(\gamma;a_2)+\cdots+n(\gamma;a_m))+0 \\ &\qquad \text{by Corollary IV.1.22}\end{aligned}$$

and the result follows. □

Example IV.7.A

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Solution. Well, $z^2 + z + 1$ has zeros at two points ω_1 and ω_2 where $|\omega_1| = |\omega_2| = 1$ (since $z^3 - 1 = (z-1)(z^2+z+1)$). So by Theorem IV.7.2 with $f(z) = z^2 + 1$, we have

$$\int_\gamma \frac{2z+1}{z^2+z+1}\,dz = 2\pi i(n(\gamma;\omega_1) + n(\gamma;\omega_2)) = 4\pi i.$$

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Let G be a region (open connected set) and suppose that f is a nonconstant analytic function on G. Then for any open set U in G, $F(U)$ is open.

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Proof. By the Open Mapping Theorem (Theorem IV.7.5), f^{-1} has inverse images of open sets open and so f^{-1} is continuous, AND $f(G) = \Omega$ is open. Since $z = f^{-1}(f(z))$, the result follows by Proposition III.2.20. □

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