

Complex Analysis

Chapter IV. Complex Integration

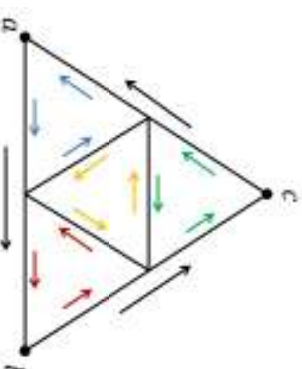
IV.8. Goursat's Theorem—Proofs of Theorems



Goursat's Theorem

Goursat's Theorem. Let G be an open set and let $f : G \rightarrow \mathbb{C}$ be a differentiable function; then f is analytic in G .

Proof. We need to show that f' is continuous on each open disk in G , so without loss of generality we assume G is itself an open disk. Let $T = [a, b, c, a]$ be a triangular path in G and let Δ be the closed set formed by T and its interior. Notice that $\partial\Delta = T$. Use the midpoints of the sides of Δ to form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ as:



Goursat's Theorem (continued 1)

Proof (continued). The $\int_T f(z) dz = \sum_{j=1}^4 \int_{T_j} f(z) dz$ where $T_j = \partial\Delta_j$.

Let $T^{(1)}$ be the T_j such that $|\int_{T^{(1)}} f(z) dz| \geq |\int_{T_j} f(z) dz|$ for $j = 1, 2, 3, 4$. The length of each T_j is half the length of T :

$\ell(T_j) = \frac{1}{2}\ell(T)$. Also, the diameter of T_j is half the diameter of T :

$\text{diam}(T_j) = \frac{1}{2}\text{diam}(T)$. By our choice of $T^{(1)}$ we have

$|\int_{T^{(1)}} f(z) dz| \leq 4 |\int_{T^{(1)}} f(z) dz|$. We now iterate this process and produce a

sequence of triangles $\{T^{(n)}\}_{n=1}^\infty$ such that $T^{(n)}$ along with its interior, $\Delta^{(n)}$, we have:

$$\Delta^{(1)} \supset \Delta^{(2)} \supset \cdots; \quad \left| \int_{T^{(n)}} f(z) dz \right| \leq 4 \left| \int_{T^{(n+1)}} f(z) dz \right|,$$

$$\ell(T^{(n+1)}) = \frac{1}{2} \ell(T^{(n)}); \quad \text{diam} \Delta^{(n+1)} = \frac{1}{2} \text{diam} \Delta^{(n)}.$$

Goursat's Theorem (continued 2)

Proof (continued). So inductively:

$$\left| \int_T f(z) dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z) dz \right| \quad (8.6)$$

$$\ell(T^{(n)}) = \left(\frac{1}{2}\right)^n \ell(T); \quad \text{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n \text{diam} \Delta.$$

Since each $\Delta^{(n)}$ is closed and $\text{diam} \Delta^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, then the nestedness of the $\Delta^{(n)}$'s implies by Cantor's Theorem (Theorem II.3.6) that $\bigcap_{n=1}^\infty \Delta^{(n)} = \{z_0\}$.

Let $\varepsilon > 0$. Since f is differentiable at z_0 , we can find $\delta > 0$ such that $B(z_0; \delta) \subset G$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Goursat's Theorem (continued 3)

Proof (continued). Alternatively,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0| \quad (8.9)$$

whenever $|z - z_0| < \delta$. Choose n such that $\text{diam} \Delta^{(n)} < (1/2)^n \text{diam} \Delta < \delta$. Since $x_0 \in \Delta^{(n)}$, this implies $\Delta^{(n)} \subset B(z_0; \delta)$. Then Cauchy's Theorem (all versions!) implies that $|\int_{T^{(n)}} z \, dz| - |\int_{T^{(n)}} dz| = 0$. Hence

$$\begin{aligned} \left| \int_{T^{(n)}} f(z) \, dz \right| &= \left| \int_{T^{(n)}} f(z) \, dz - \underbrace{\int_{T^{(n)}} f(z_0) \, dz}_0 \right. \\ &\quad \left. - \underbrace{\int_{T^{(n)}} f'(z_0) z \, dz}_0 + \underbrace{\int_{T^{(n)}} f'(z_0) z_0 \, dz}_0 \right| \end{aligned}$$

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Goursat's Theorem (continued 5)

Goursat's Theorem. Let G be an open set and let $f : G \rightarrow \mathbb{C}$ be a differentiable function; then f is analytic in G .

Proof (continued). Next,

$$\begin{aligned} \left| \int_T f(z) \, dz \right| &\leq 4^n \left| \int_{T^{(n)}} f(z) \, dz \right| \text{ by (8.6)} \\ &< 4^n \varepsilon \left(\frac{1}{4^n} \right) \text{diam} \Delta \ell(T) \\ &= \varepsilon \text{diam} \Delta \ell(T). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\text{diam} \Delta, \ell(T)$ are fixed, it follows that $\int_T f(z) \, dz = 0$. So by Morera's Theorem, f is analytic on G . \square

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Goursat's Theorem (continued 4)

Proof (continued).

$$\begin{aligned} \left| \int_{T^{(n)}} f(z) \, dz \right| &= \left| \int_{T^{(n)}} (f(z) - f(z_0) - f'(z_0)(z - z_0)) \, dz \right| \\ &\leq \int_{T^{(n)}} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \, |dz| \\ &\leq \varepsilon \int_{T^{(n)}} |z - z_0| \, |dz| \text{ by (8.9)} \\ &\leq \varepsilon \text{diam} \Delta^{(n)} \ell(T^{(n)}) \\ &= \varepsilon \left(\frac{1}{2} \right)^n \text{diam} \Delta \left(\frac{1}{2} \right)^n \ell(T) \\ &= \varepsilon \left(\frac{1}{4} \right)^n \text{diam} \Delta \ell(T). \end{aligned}$$

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