Complex Analysis

Chapter IV. Complex Integration

IV.8. Goursat's Theorem—Proofs of Theorems

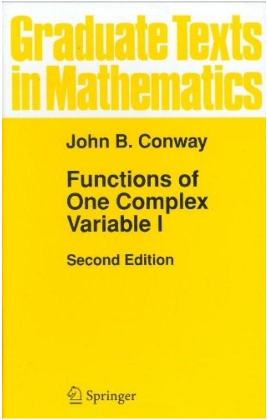


Table of contents

Goursat's Theorem

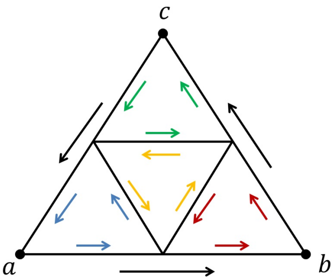
Goursat's Theorem. Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic in G.

Proof. We need to show that f' is continuous on each open disk in G, so without loss of generality we assume G is itself an open disk. Let $T = [a, b, c, a]$ be a triangular path in G and let Δ be the closed set formed by T and its interior. Notice that $\partial\Delta = T$. Use the midpoints of the sides of Δ to form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ as:

Goursat's Theorem

Goursat's Theorem. Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic in G.

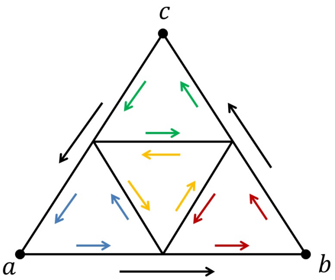
Proof. We need to show that f' is continuous on each open disk in G, so without loss of generality we assume G is itself an open disk. Let $T = [a, b, c, a]$ be a triangular path in G and let Δ be the closed set formed by T and its interior. Notice that $\partial\Delta = T$. Use the midpoints of the sides of Δ to form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ as:



Goursat's Theorem

Goursat's Theorem. Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic in G.

Proof. We need to show that f' is continuous on each open disk in G, so without loss of generality we assume G is itself an open disk. Let $T = [a, b, c, a]$ be a triangular path in G and let Δ be the closed set formed by T and its interior. Notice that $\partial\Delta = T$. Use the midpoints of the sides of Δ to form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ as:



Goursat's Theorem (continued 1)

Proof (continued). The $\int_T f(z)\,dz = \sum_{j=1}^4 \int_{T_j} f(z)\,dz$ where $T_j = \partial\Delta_j$. Let $T^{(1)}$ be the T_j such that $\left|\int_{T^{(1)}} f(z)\,dz\right| \geq \left|\int_{T_j} f(z)\,dz\right|$ for $j = 1, 2, 3, 4$. The length of each T_j is half the length of T: $\ell(T_j) = \frac{1}{2}\ell(T)$. Also, the diameter of T_j is half the diameter of T: $\text{diam}(T_j) = \frac{1}{2}\text{diam}(T)$. By our choice of $T^{(1)}$ we have $\left|\int_T f(z)\,dz\right| \leq 4\left|\int_{T^{(1)}} f(z)\,dz\right|$. We now iterate this process and produce a sequence of triangles $\{T^{(n)}\}_{n=1}^\infty$ such that $T^{(n)}$ along with its interior, $\Delta^{(n)}$, we have:

$$\Delta^{(1)} \supset \Delta^{(2)} \supset \cdots; \ \left|\int_{T^{(n)}} f(z)\,dz\right| \leq 4\left|\int_{T^{(n+1)}} f(z)\,dz\right|,$$

$$\ell(T^{(n+1)}) = \frac{1}{2}\ell(T^{(n)}); \ \text{diam}\Delta^{(n+1)} = \frac{1}{2}\text{diam}\Delta^{(n)}.$$

Goursat's Theorem (continued 1)

Proof (continued). The $\int_T f(z)\,dz = \sum_{j=1}^4 \int_{T_j} f(z)\,dz$ where $T_j = \partial\Delta_j$. Let $T^{(1)}$ be the T_j such that $\left|\int_{T^{(1)}} f(z)\,dz\right| \geq \left|\int_{T_j} f(z)\,dz\right|$ for $j = 1, 2, 3, 4$. The length of each T_j is half the length of T: $\ell(T_j) = \frac{1}{2}\ell(T)$. Also, the diameter of T_j is half the diameter of T: $\text{diam}(T_j) = \frac{1}{2}\text{diam}(T)$. By our choice of $T^{(1)}$ we have $\left|\int_T f(z)\,dz\right| \leq 4\left|\int_{T^{(1)}} f(z)\,dz\right|$. We now iterate this process and produce a sequence of triangles $\{T^{(n)}\}_{n=1}^\infty$ such that $T^{(n)}$ along with its interior, $\Delta^{(n)}$, we have:

$$\Delta^{(1)} \supset \Delta^{(2)} \supset \cdots; \quad \left|\int_{T^{(n)}} f(z)\,dz\right| \leq 4\left|\int_{T^{(n+1)}} f(z)\,dz\right|,$$

$$\ell(T^{(n+1)}) = \frac{1}{2}\ell(T^{(n)}); \ \text{diam}\Delta^{(n+1)} = \frac{1}{2}\text{diam}\Delta^{(n)}.$$

Goursat's Theorem (continued 2)

Proof (continued). So inductively:

$$\left|\int_T f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \qquad (8.6)$$

$$\ell(T^{(n)}) = \left(\frac{1}{2}\right)^n \ell(T);\ \text{diam}\Delta^{(n)} = \left(\frac{1}{2}\right)^n \text{diam}\Delta.$$

Since each $\Delta^{(n)}$ is closed and diam$\Delta^{(n)} \to 0$ as $n \to 0$, then the nestedness of the $\Delta^{(n)}$'s implies by Cantor's Theorem (Theorem II.3.6) that $\cap_{n=1}^{\infty}\Delta^{(n)} = \{z_0\}$.

Goursat's Theorem (continued 2)

Proof (continued). So inductively:

$$\left|\int_T f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \qquad (8.6)$$

$$\ell(T^{(n)}) = \left(\frac{1}{2}\right)^n \ell(T);\ \text{diam}\Delta^{(n)} = \left(\frac{1}{2}\right)^n \text{diam}\Delta.$$

Since each $\Delta^{(n)}$ is closed and $\text{diam}\Delta^{(n)} \to 0$ as $n \to 0$, then the nestedness of the $\Delta^{(n)}$'s implies by Cantor's Theorem (Theorem II.3.6) that $\cap_{n=1}^{\infty}\Delta^{(n)} = \{z_0\}$.

Let $\varepsilon > 0$. Since f is differentiable at z_0, we can find $\delta > 0$ such that $B(z_0;\delta) \subset G$ and

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Goursat's Theorem (continued 2)

Proof (continued). So inductively:

$$\left|\int_T f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \qquad (8.6)$$

$$\ell(T^{(n)}) = \left(\frac{1}{2}\right)^n \ell(T);\ \text{diam}\Delta^{(n)} = \left(\frac{1}{2}\right)^n \text{diam}\Delta.$$

Since each $\Delta^{(n)}$ is closed and $\text{diam}\Delta^{(n)} \to 0$ as $n \to 0$, then the nestedness of the $\Delta^{(n)}$'s implies by Cantor's Theorem (Theorem II.3.6) that $\cap_{n=1}^{\infty}\Delta^{(n)} = \{z_0\}$.

Let $\varepsilon > 0$. Since f is differentiable at z_0, we can find $\delta > 0$ such that $B(z_0;\delta) \subset G$ and

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$.

Goursat's Theorem (continued 3)

Proof (continued). Alternatively,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon |z - z_0| \qquad (8.9)$$

whenever $|z - z_0| < \delta$. Choose n such that $\text{diam}\Delta^{(n)} < (1/2)^n \text{diam}\Delta < \delta$. Since $x_0 \in \Delta^{(n)}$, this implies $\Delta^{(n)} \subset B(z_0; \delta)$. Then Cauchy's Theorem (all versions!) implies that $\left|\int_{T^{(n)}} z\, dz\right| - \left|\int_{T^{(n)}} dz\right| = 0$. Hence

$$\left|\int_{T^{(n)}} f(z)\, dz\right| = \left|\int_{T^{(n)}} f(z)\, dz - \underbrace{\int_{T^{(n)}} f(z_0)\, dz}_{0}\right.$$

$$\left. - \underbrace{\int_{T^{(n)}} f'(z_0) z\, dz}_{0} + \underbrace{\int_{T^{(n)}} f'(z_0) z_0\, dz}_{0}\right|$$

Goursat's Theorem (continued 3)

Proof (continued). Alternatively,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon|z - z_0| \qquad (8.9)$$

whenever $|z - z_0| < \delta$. Choose n such that $\text{diam}\Delta^{(n)} < (1/2)^n \text{diam}\Delta < \delta$. Since $x_0 \in \Delta^{(n)}$, this implies $\Delta^{(n)} \subset B(z_0; \delta)$. Then Cauchy's Theorem (all versions!) implies that $\left|\int_{T^{(n)}} z\, dz\right| - \left|\int_{T^{(n)}} dz\right| = 0$. Hence

$$\left|\int_{T^{(n)}} f(z)\, dz\right| = \left|\int_{T^{(n)}} f(z)\, dz - \underbrace{\int_{T^{(n)}} f(z_0)\, dz}_{0}\right.$$

$$\left. - \underbrace{\int_{T^{(n)}} f'(z_0) z\, dz}_{0} + \underbrace{\int_{T^{(n)}} f'(z_0) z_0\, dz}_{0}\right|$$

Goursat's Theorem (continued 4)

Proof (continued).

$$\begin{aligned}
\left|\int_{T^{(n)}} f(z)\,dz\right| &= \left|\int_{T^{(n)}} (f(z)-f(z_0)-f'(z_0)(z-z_0))\,dz\right| \\
&\le \int_{T^{(n)}} |f(z)-f(z_0)-f'(z_0)(z-z_0)|\,|dz| \\
&\le \varepsilon \int_{T^{(n)}} |z-z_0|\,|dz| \text{ by (8.9)} \\
&\le \varepsilon\,\mathrm{diam}\Delta^{(n)}\,\ell(T^{(n)}) \\
&= \varepsilon \left(\frac{1}{2}\right)^n \mathrm{diam}\Delta \left(\frac{1}{2}\right)^n \ell(T) \\
&= \varepsilon \left(\frac{1}{4}\right)^n \mathrm{diam}\Delta\,\ell(T).
\end{aligned}$$

Goursat's Theorem (continued 5)

Goursat's Theorem. Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic in G.

Proof (continued). Next,

$$\begin{aligned} \left| \int_T f(z)\, dz \right| &\le 4^n \left| \int_{T^{(n)}} f(z)\, dz \right| \text{ by (8.6)} \\ &< 4^n \varepsilon \left(\frac{1}{4^n} \right) \operatorname{diam}\Delta\, \ell(T) \\ &= \varepsilon \operatorname{diam}\Delta\, \ell(T). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\operatorname{diam}\Delta$, $\ell(T)$ are fixed, it follows that $\int_T f(z)\, dz = 0$. So by Morera's Theorem, f is analytic on G. □

Goursat's Theorem (continued 5)

Goursat's Theorem. Let G be an open set and let $f : G \to \mathbb{C}$ be a differentiable function; then f is analytic in G.

Proof (continued). Next,

$$\begin{aligned} \left| \int_T f(z)\,dz \right| &\leq 4^n \left| \int_{T^{(n)}} f(z)\,dz \right| \text{ by (8.6)} \\ &< 4^n \varepsilon \left(\frac{1}{4^n} \right) \text{diam}\Delta\,\ell(T) \\ &= \varepsilon\,\text{diam}\Delta\,\ell(T). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and diamΔ, $\ell(T)$ are fixed, it follows that $\int_T f(z)\,dz = 0$. So by Morera's Theorem, f is analytic on G. □