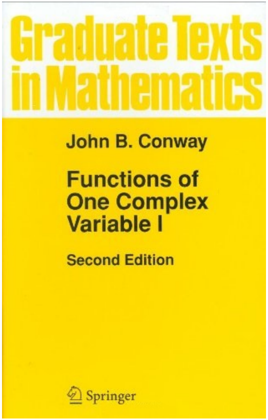
Complex Analysis

Chapter IX. Analytic Continuation and riemann Surfaces

IX.2. Analytic Continuation Along a Path—Proofs of Theorems



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Proof. We will show that the set $T = \{t \in [0,1] \mid [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$ is both open and closed in $[0,1]$. The only subsets of $[0,1]$ that are both open and closed in $[0,1]$ are $\varnothing$ and $[0,1]$ (because $[0,1]$ is connected). Since $0 \in T$ then $T \neq \varnothing$. So we will conclude that $T = [0,1]$ and so $1 \in T$ and the conclusion will follow.

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To prove that T is open, let $t \in T$ where $t \not\in \{0,1\}$. By the definition of analytic continuation for the two analytic continuations) such that for $|s-t| < \delta$, $\gamma(s) \in D_t \cap B_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, [g_s]_{\gamma(s)} = [g_t]_{\sigma(s)}. \qquad (2.5)$$

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Proposition IX.2.4 (continued 1)

Proof (continued). Let H be a connected subset of $D_t \cap B_t$ which contains $\gamma(s)$ and $\gamma(t)$ (notice the segment of the path γ between $\gamma(t)$ and $\gamma(s)$ is connected so such a component exists). Since $t \in T$, by definition of set T we have that the germs of f and g at $\gamma(s)$ are equal: $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$. That is, $f_t(z) = g_t(z)$ for all $z \in H$ (see Note IX.2.A). Hence $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ for all $\gamma(s)$ in $D_t \cap B_t$. So by (2.5), for all $|s-t| < \delta$ we have $[f_s]_{\gamma(s)} - [f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} = [g_s]_{\gamma(s)}$. So, by the definition of set T, $s \in T$ for all $|s-t| < \delta$; or $(t-\delta, t+\delta) \subset T$. If $t = 0$, this argument can be modified to show that there is $\delta > 0$ such that $[0, \delta) \subset T$. If $t = 1$, the Proposition immediately holds. So T is open in $[0,1]$.

Proposition IX.2.4 (continued 1)

Proof (continued). Let H be a connected subset of $D_t \cap B_t$ which contains $\gamma(s)$ and $\gamma(t)$ (notice the segment of the path γ between $\gamma(t)$ and $\gamma(s)$ is connected so such a component exists). Since $t \in T$, by definition of set T we have that the germs of f and g at $\gamma(s)$ are equal: $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$. That is, $f_t(z) = g_t(z)$ for all $z \in H$ (see Note IX.2.A). Hence $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ for all $\gamma(s)$ in $D_t \cap B_t$. So by (2.5), for all $|s - t| < \delta$ we have $[f_s]_{\gamma(s)} - [f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} = [g_s]_{\gamma(s)}$. So, by the definition of set T, $s \in T$ for all $|s - t| < \delta$; or $(t - \delta, t + \delta) \subset T$. If $t = 0$, this argument can be modified to show that there is $\delta > 0$ such that $[0, \delta) \subset T$. If $t = 1$, the Proposition immediately holds. So T is open in $[0, 1]$.

To show that T is closed in $[0, 1]$, let t be a limit point of T. Again choose $\delta > 0$ so that $\gamma(s) \in D_t \cap B_t$ an d(2.5) is satisfied whenever $|s - t| < \delta$. Since t is a limit point of T there is a point $s \in T$ where $|s - t| < \delta$. Let G be a region such that $\gamma((t - \delta, t + \delta)) \subset G \subset D_t \cap B_t$; in particular, $\gamma(s) \in G$.

Proposition IX.2.4 (continued 1)

Proof (continued). Let H be a connected subset of $D_t \cap B_t$ which contains $\gamma(s)$ and $\gamma(t)$ (notice the segment of the path γ between $\gamma(t)$ and $\gamma(s)$ is connected so such a component exists). Since $t \in T$, by definition of set T we have that the germs of f and g at $\gamma(s)$ are equal: $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$. That is, $f_t(z) = g_t(z)$ for all $z \in H$ (see Note IX.2.A). Hence $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ for all $\gamma(s)$ in $D_t \cap B_t$. So by (2.5), for all $|s - t| < \delta$ we have $[f_s]_{\gamma(s)} - [f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} = [g_s]_{\gamma(s)}$. So, by the definition of set T, $s \in T$ for all $|s - t| < \delta$; or $(t - \delta, t + \delta) \subset T$. If $t = 0$, this argument can be modified to show that there is $\delta > 0$ such that $[0, \delta) \subset T$. If $t = 1$, the Proposition immediately holds. So T is open in $[0, 1]$.

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Proposition IX.2.4 (continued 2)

Proof (continued). Then by the definition of set T, $[f_s]_{\gamma(s)} = [g_x]_{\gamma(s)}$ and so

$$f_s(z) = g_s(z) \text{ for all } z \in G. \qquad (*)$$

Since (2.5) is satisfied for $|s - t| < \delta$ then $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$ and $[g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}$; that is

$$f_s(z) = f_t(z) \text{ and } g_s(z) = g_t(z) \text{for all } z \in G. \qquad (**)$$

Combining $(*)$ and $(**)$ gives $f_t(z) = f_s(z) = g_s(z) = g_t(z)$ for all $z \in G$. Since G (a region) satisfies $G \subset D_T \cap B_t$ then G has a limit point in $D_t \cap B_t$ (and both f_t and g_t are analytic on $D_t \cap B_t$) then by Corollary IV.3.8 $f_t(z) = g_t(z)$ for all $z \in D_t \cap B_t$. Since $\gamma(t) \in D_t \cap B_t$ we have $[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}$. That is (by definition of T), $t \in T$ and so T is closed in $[0, 1]$.

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Since T is nonempty and both open and closed in $[0, 1]$, then $T = [0, 1]$. So $1 \in T$ and (by definition of T), $[f_1]_{\gamma(1)} = [g_1]_{\gamma(1)}$ or $[f_1]_b = [g_1]_b$. □

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Since T is nonempty and both open and closed in $[0,1]$, then $T = [0,1]$. So $1 \in T$ and (by definition of T), $[f_1]_{\gamma(1)} = [g_1]_{\gamma(1)}$ or $[f_1]_b = [g_1]_b$. □