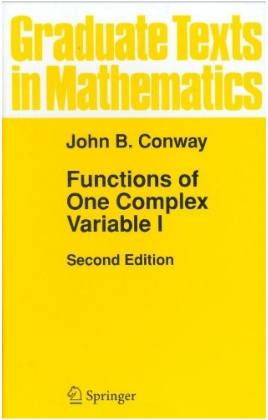
Complex Analysis

Chapter IX. Analytic Continuation and Riemann Surfaces

IX.3. Monodromy Theorem—Proofs of Theorems



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Lemma IX.3.1

Lemma IX.3.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a point and let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ be an analytic continuation along γ. For $0 \leq t \leq 1$ let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(T) = \infty$ or $T : [0,1] \to (0,\infty)$ is continuous.

Proof. If $R(t) = \infty$ for some $t \in [0,1]$ then f_t can be extended to an entire function f (the power series representation of f_t centered at $\gamma(t)$). By the definition of analytic continuation we can conclude $[f_s]_{\gamma(s)} = [f_t]_{\gamma(t)}$ for all $s, t \in [0,1]$ (for example, for each $t \in [0,1]$ consider the open relative to $[0,1]$ set $(t-\delta, t+\delta) \cap [0,1]$ where δ is as given in the definition of analytic continuation; for the resulting open cover of $[0,1]$, extract a finite subcover and "walk" across γ picking up $[f_s]_{\gamma(s)} = [f_t]_{\gamma(t)}$ for each of the finite segments of γ).

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Lemma IX.3.1 (continued 1)

Proof (continued). Suppose $R(t) < \infty$ for all $t \in [0,1]$. Fix t and let $\tau = \gamma(t)$. Let $f_t = \sum_{n=0}^{\infty} \tau_n (z-\tau)^n$. By the definition of analytic continuation there is $\delta_1 > 0$ such that $|s-t| < \delta_1$ implies $\gamma(s) \in D_t \cap B(\tau; R(t))$ (we have $\delta > 0$ such that $\gamma(s) \in D_t$ for $|s-t| > | < \delta$ and there is $\delta' > 0$ such that $|\gamma(s) - \gamma(t)| = |\gamma(s) - \tau| < R(t)$ since γ is continuous at t; let $\delta_1 = \min\{\delta, \delta'\}$) and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. Fix $s \in [0,1]$ with $|s-t| < \delta_1$ and let $\sigma = \gamma(s)$. Now f_t can be extended to an analytic function on all of $B(\tau; R(t))$ (namely, the power series for f_t given above). Since $f_s(z) = f_t(z)$ on some neighborhood of $\sigma = \gamma(s)$, then f_s can be extended so that it is analytic on $B(\tau; R(t)) \cup D_s$. So let f_s have power series expansion $f_s(z) = \sum_{n=0}^{\infty} \sigma_n (z-\sigma)^n$ about $z = \sigma$ with radius of convergence $R(s)$.

Lemma IX.3.1 (continued 1)

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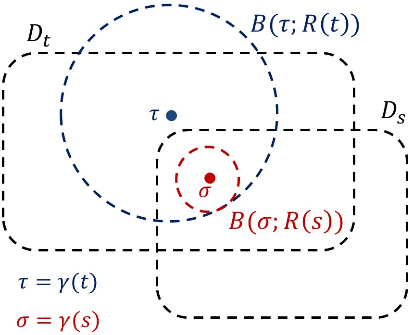
Lemma IX.3.1 (continued 2)

Proof (continued). We have:



Since $f_t(z) = f_s(z)$ on some neighborhood of γ and f_t is analytic on $B(\tau; R(t))$ then the power series for f_s about σ must have radius of convergence that at least reaches the boundary of $B(\tau; R(t))$ as shown above. That is, $R(s)$ is at least as big as the distance from σ to the circle $|z - \tau| = R(t)$. So $R(s) \geq d(\sigma, \{z \mid |z - \tau| = R(t)\}) = R(t) - |\tau - \sigma|$.

Lemma IX.3.1 (continued 3)

Lemma IX.3.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a point and let $\{(f_t, D_t) \mid 0 \le t \le 1\}$ be an analytic continuation along γ. For $0 \le t \le 1$ let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(T) = \infty$ or $T : [0,1] \to (0, \infty)$ is continuous.

Proof (continued). But this gives $R(t) - R(s) \le |\tau - \sigma| = |\gamma(t) - \gamma(s)|$. A similar argument (interchanging the roles of s and t) gives that $R(s) - R(t) \le |\gamma(t) - \gamma(s)|$. Hence $|R(t) - R(s)| \le |\gamma(t) - \gamma(s)|$ for $|t - s| < \delta_1$. Let $t \in [0,1]$ and let $\varepsilon > 0$. Since γ is continuous at t there is $\delta_2 > 0$ such that $|t - s| < \delta_2$ implies $|\gamma(t) - \gamma(s)| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $|t - s| < \delta$ we have $|R(t) - R(s)| \le |\gamma(t) - \gamma(s)| < \varepsilon$ and so R is continuous at $T \in [0,1]$. Since t is an arbitrary element of $[0,1]$, R is continuous on $[0,1]$. □

Lemma IX.3.1 (continued 3)

Lemma IX.3.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a point and let $\{(f_t, D_t) \mid 0 \le t \le 1\}$ be an analytic continuation along γ. For $0 \le t \le 1$ let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(T) = \infty$ or $T : [0,1] \to (0, \infty)$ is continuous.

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Lemma IX.3.2

Lemma IX.3.2. Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) \mid 0 \le t \le 1\}$ be an analytic continuation along path γ. There is a number $\varepsilon > 0$ such that if $\sigma : [0,1] \to \mathbb{C}$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \varepsilon$ for all $t \in [0,1]$, and if $\{(g_t, B_t) \mid 0 \le t \le 1\}$ is any continuation along σ with $[g_0]_a = [f_0]_a$; then $[g_1]_a = [f_1]_b$.

Proof. For $0 \le t \le 1$, let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. By Lemma IX.3.1, either $R(t) \equiv 0$ or $R(t)$ is continuous and (finite) positive valued. If $R(t) \equiv \infty$ then there is entire function f such that $f \equiv f_t$ on D_t and $f = g_t$ on B_t. In particular, $f = f - 1$ on D_1, $f = g_1$ on B_1, and $b \in D_1 \cap B_1$. Since $D_1 \cap B - 1$ is an open set then $f(z) = f_1(z) = g_1(z)$ on some neighborhood of 1 and so $[g_1]_b = [f_1]_b$. So we now assume $R(t) < \infty$ for all $t \in [0,1]$, $R(t) > 0$, and that R is continuous.

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Lemma IX.3.2 (continued 1)

Proof (continued). Let

$$0 < \varepsilon < \frac{1}{2}\{R(t) \mid 0 \le t \le 1\} \qquad (3.3)$$

and suppose that σ and $\{(g_t, B_t) \mid 0 \le t \le 1\}$ is as hypothesized. Since, in the definition of analytic continuation, "the sets D_t in the definition can be enlarged or diminished without affecting the fact that there is a continuation" (Conway, page 214), we can assume that D_t is a disk centered at $\gamma(t)$ with radius $R(t)$. Similarly, assume B_t is a disk centered at $\sigma(t)$ on which g_t is analytic.

Since $|\sigma(t) - \gamma(t)| < \varepsilon$ by hypothesis and $\varepsilon < R(t)$ by choice of ε then $|\sigma(t) - \gamma(t)| < R(t)$, so $\sigma(t) \in D_t$, and, since $\sigma(t) \in B_t$ by definition of B_t, then $\sigma(t) \in B_t \cap D_t$. So it makes sense to ask whether $g_t(z) = f_t(z)$ for all $z \in B_t \cap D_t$. If we can show this is the case for $t = 1$, then the claim follows. Define $T = \{t \in [0,1] \mid f_t(z) = g_t(z) \text{ for all } z \in B_t \cap D_t\}$.

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Lemma IX.3.2 (continued 2)

Proof (continued). Similar to the proof of Proposition IX.2.4, we show that T is a nonempty open and closed subset of $[0,1]$ and conclude that $T = [0,1]$; that is, $1 \in T$. By hypothesis, $[g_0]_a = [f_0]_a$ and so $0 \in T$ and $T \neq \varnothing$. To show that T is open, fix $t \in T$ and choose $\delta > 0$ such that

$$\begin{cases} |\gamma(s) - \gamma(t)| < \varepsilon, & [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, \\ |\sigma(s) - \sigma(t)| < \varepsilon, & [g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}, \quad \text{and} \\ \sigma(s) \in B_t \end{cases} \qquad (3.4)$$

whenever $|s - t| < \delta$. Such a $\delta > 0$ exists for the first two conditions by the definition of analytic continuation and the third condition holds since $\sigma(t) \in B_t \cap D_t$ for all $t \in [0,1]$ (as shown above) and σ is continuous (so $\sigma(s) \in B_t$ for s "sufficiently close" to t).

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Lemma IX.3.2 (continued 3)

Proof (continued). Also, by (3.3)

$$|\sigma(s) - \gamma(t)| = |\sigma(s) - \gamma(s) + \gamma(s) - \gamma(t)| \leq |\sigma(s) - \gamma(s)| + |\gamma(s) - \gamma(t)|$$

$$= \varepsilon + \varepsilon = 2\varepsilon < R(t).$$

So $\sigma(s) \in D_t$ (since D_t is a disk of radius $R(t)$ centered at $\gamma(t)$). By (3.4) $\sigma(s) \in B_t$ and $\gamma(s) \in B_s$ by the definition of B_s, so $\sigma(s) \in B_s \cap B_t \cap D_s \cap D_t \equiv G$. Since $t \in T$ by hypothesis, the definition of T gives that $f_t(z) = g_t(z)$ for all $z \in G$. From (3.4), $[f_2]_{\gamma(s)} = [f_t]_{\gamma(s)}$ and $[g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}|$, or $f_s(z) = f_t(z)$ for all $z \in D_s$ and $g_s(z) = g_t(z)$ for all $z \in B_s$. Hence $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all $z \in G$.

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Lemma IX.3.2 (continued 3)

Proof (continued). Also, by (3.3)

$$|\sigma(s) - \gamma(t)| = |\sigma(s) - \gamma(s) + \gamma(s) - \gamma(t)| \leq |\sigma(s) - \gamma(s)| + |\gamma(s) - \gamma(t)|$$

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So $\sigma(s) \in D_t$ (since D_t is a disk of radius $R(t)$ centered at $\gamma(t)$). By (3.4) $\sigma(s) \in B_t$ and $\gamma(s) \in B_s$ by the definition of B_s, so $\sigma(s) \in B_s \cap B_t \cap D_s \cap D_t \equiv G$. Since $t \in T$ by hypothesis, the definition of T gives that $f_t(z) = g_t(z)$ for all $z \in G$. From (3.4), $[f_2]_{\gamma(s)} = [f_t]_{\gamma(s)}$ and $[g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}|$, or $f_s(z) = f_t(z)$ for all $z \in D_s$ and $g_s(z) = g_t(z)$ for all $z \in B_s$. Hence $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all $z \in G$. But since G has a limit point in $B_s \cap D_s$ (all of these sets are nonempty intersections of open sets) and $f_s(z) = f_t(z) = g_t(z) = g_s(t)$ for all $z \in B_s \cap D_s$ and so $s \in T$. That is, $(t - \delta, t + \delta) \subset T$ and so T is open in $[0, 1]$.

In Exercise IX.3.1, it is shown that T is closed in $[0, 1]$. The claim then follows. □

Lemma IX.3.2 (continued 3)

Proof (continued). Also, by (3.3)

$$|\sigma(s)-\gamma(t)| = |\sigma(s)-\gamma(s)+\gamma(s)-\gamma(t)| \leq |\sigma(s)-\gamma(s)|+|\gamma(s)-\gamma(t)|$$

$$= \varepsilon + \varepsilon = 2\varepsilon < R(t).$$

So $\sigma(s) \in D_t$ (since D_t is a disk of radius $R(t)$ centered at $\gamma(t)$). By (3.4) $\sigma(s) \in B_t$ and $\gamma(s) \in B_s$ by the definition of B_s, so $\sigma(s) \in B_s \cap B_t \cap D_s \cap D_t \equiv G$. Since $t \in T$ by hypothesis, the definition of T gives that $f_t(z) = g_t(z)$ for all $z \in G$. From (3.4), $[f_2]_{\gamma(s)} = [f_t]_{\gamma(s)}$ and $[g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}|$, or $f_s(z) = f_t(z)$ for all $z \in D_s$ and $g_s(z) = g_t(z)$ for all $z \in B_s$. Hence $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all $z \in G$. But since G has a limit point in $B_s \cap D_s$ (all of these sets are nonempty intersections of open sets) and $f_s(z) = f_t(z) = g_t(z) = g_s(t)$ for all $z \in B_s \cap D_s$ and so $s \in T$. That is, $(t-\delta, t+\delta) \subset T$ and so T is open in $[0,1]$.

In Exercise IX.3.1, it is shown that T is closed in $[0,1]$. The claim then follows. □

Theorem IX.3.6

Theorem IX.3.6. Monodromy Theorem.
Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then $[f_1]_b = [g_1]_b$.

Proof. Since γ_0 and γ_1 are fixed end point homotopic in G then there is a continuous $\Gamma : [0, 1] \times [0, 1] \to G$ such that

$$\Gamma(t,) = \gamma_0(t), \ \ \Gamma(t, 1) = \gamma_1(t),$$

$$\Gamma(0, u) = a, \ \ \Gamma(1, u) = b$$

for all $t, u \in [0, 1]$.

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for all $t, u \in [0,1]$. Fix $u \in [0,1]$ and consider γ_u defined as $\gamma(u) = \Gamma(t,u)$ (a path from a to b). Since (f, D) admits unrestricted continuation on G by hypothesis then there is an analytic continuation $\{(h_{t,u}, D_{t,u}) \mid 0 \le t \le 1\}$ of (f, D) along γ_u.

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Theorem IX.3.6 (continued 1)

Proof (continued). By Proposition IX.2.4 the two analytic continuations $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,0}, D_{t,0}) \mid 0 \leq t \leq 1\}$ along γ_0 yield $[f_1]_b = [h_{1,0}]_b$, and the two analytic continuations $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,1}, D_{t,1}) \mid 0 \leq t \leq 1\}$ along γ_1 yield $[g_1]_b = [h_{t,1}]_b$. So the claim will follow if we show that $[h_{1,0}]_b = [h_{1,1}]_b$. Consider the set $U = \{u \in [0,1] \mid [h_{1,u}]_b = [h_{1,0}]_b\}$. We now show that U is nonempty and both open and closed in $[0,1]$ (so that, since $[0,1]$ is connected, $U = [0,1]$, $1 \in U$, and the claim follows). Now $0 \in U$ so $U \neq \varnothing$.

Theorem IX.3.6 (continued 1)

Proof (continued). By Proposition IX.2.4 the two analytic continuations $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,0}, D_{t,0}) \mid 0 \leq t \leq 1\}$ along γ_0 yield $[f_1]_b = [h_{1,0}]_b$, and the two analytic continuations $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,1}, D_{t,1}) \mid 0 \leq t \leq 1\}$ along γ_1 yield $[g_1]_b = [h_{t,1}]_b$. So the claim will follow if we show that $[h_{1,0}]_b = [h_{1,1}]_b$. Consider the set $U = \{u \in [0,1] \mid [h_{1,u}]_b = [h_{1,0}]_b\}$. We now show that U is nonempty and both open and closed in $[0,1]$ (so that, since $[0,1]$ is connected, $U = [0,1]$, $1 \in U$, and the claim follows). Now $0 \in U$ so $U \neq \varnothing$.

Next we establish:

Claim 3.7. For $u \in [0,1]$ there is $\delta > 0$ such that if $|u - v| < \delta$ then $[h_{1,u}]_b = [h_{1,v}]_b$.

Theorem IX.3.6 (continued 1)

Proof (continued). By Proposition IX.2.4 the two analytic continuations $\{(f_t, D_t) \mid 0 \le t \le 1\}$ and $\{(h_{t,0}, D_{t,0}) \mid 0 \le t \le 1\}$ along γ_0 yield $[f_1]_b = [h_{1,0}]_b$, and the two analytic continuations $\{(g_t, B_t) \mid 0 \le t \le 1\}$ and $\{(h_{t,1}, D_{t,1}) \mid 0 \le t \le 1\}$ along γ_1 yield $[g_1]_b = [h_{t,1}]_b$. So the claim will follow if we show that $[h_{1,0}]_b = [h_{1,1}]_b$. Consider the set $U = \{u \in [0,1] \mid [h_{1,u}]_b = [h_{1,0}]_b\}$. We now show that U is nonempty and both open and closed in $[0,1]$ (so that, since $[0,1]$ is connected, $U = [0,1]$, $1 \in U$, and the claim follows). Now $0 \in U$ so $U \ne \varnothing$.

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For a fixed $u \in [0,1]$, Lemma IX.3.2 implies that there is $\varepsilon > 0$ such that if σ is any path from a to b with $|\gamma_u(t) - \sigma(t)| < \varepsilon$ for all $t \in [0,1]$, and if $\{(k_t, E_t) \mid 0 \le t \le 1\}$ is any analytic continuation of (f, D) along σ, then

$$[h_{1,u}]_b = [k_1]_b \qquad (3.8)$$

Theorem IX.3.6 (continued 1)

Proof (continued). By Proposition IX.2.4 the two analytic continuations $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,0}, D_{t,0}) \mid 0 \leq t \leq 1\}$ along γ_0 yield $[f_1]_b = [h_{1,0}]_b$, and the two analytic continuations $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ and $\{(h_{t,1}, D_{t,1}) \mid 0 \leq t \leq 1\}$ along γ_1 yield $[g_1]_b = [h_{t,1}]_b$. So the claim will follow if we show that $[h_{1,0}]_b = [h_{1,1}]_b$. Consider the set $U = \{u \in [0,1] \mid [h_{1,u}]_b = [h_{1,0}]_b\}$. We now show that U is nonempty and both open and closed in $[0,1]$ (so that, since $[0,1]$ is connected, $U = [0,1]$, $1 \in U$, and the claim follows). Now $0 \in U$ so $U \neq \varnothing$.

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For a fixed $u \in [0,1]$, Lemma IX.3.2 implies that there is $\varepsilon > 0$ such that if σ is any path from a to b with $|\gamma_u(t) - \sigma(t)| < \varepsilon$ for all $t \in [0,1]$, and if $\{(k_t, E_t) \mid 0 \leq t \leq 1\}$ is any analytic continuation of (f, D) along σ, then

$$[h_{1,u}]_b = [k_1]_b \qquad (3.8)$$

Theorem IX.3.6 (continued 2)

Proof (continued). Now Γ is continuous and $[0,1] \times [0,1]$ is compact, to Γ is uniformly continuous by Theorem II.5.15, so there is a $\delta > 0$ such that if $|u - v| < \delta$ then $|\gamma_u(t) - \gamma_v(t)| = |\Gamma(t,u) - \Gamma(t,v)| < \varepsilon$ for all $t \in [0,1]$. So with $\sigma = \gamma_v$, we have by (3.8) that $[h_{1,u}]_b = [k_1]_b$; with $\sigma = \gamma_u$, we have by (3.8) that $[h_{1,v}]_b = [k_1]_b$. Therefore $[h_{1,u}]_b = [h_{1,v}]_b$ and Claim 3.7 holds.

Suppose $u \in U$ and let $\delta > 0$ be the number given by Claim 3.7. Since $u \in U$ then by the definition of U, $[h_{1,u}]_b = [h_{1,0}]_b$. By Claim 3.7, for $|u - v| < \delta$ we have $[h_{1,u}]_b = [h_{1,v}]_b$. So for $|u - v| < \delta$ we have $[h_{1,v}]_b = [h_{1,0}]_b$ and so $v \in U$; that is, $(u - \delta, u + \delta) \subset U$. So U is open in $[0,1]$.

Theorem IX.3.6 (continued 2)

Proof (continued). Now Γ is continuous and $[0,1] \times [0,1]$ is compact, to Γ is uniformly continuous by Theorem II.5.15, so there is a $\delta > 0$ such that if $|u - v| < \delta$ then $|\gamma_u(t) - \gamma_v(t)| = |\Gamma(t,u) - \Gamma(t,v)| < \varepsilon$ for all $t \in [0,1]$. So with $\sigma = \gamma_v$, we have by (3.8) that $[h_{1,u}]_b = [k_1]_b$; with $\sigma = \gamma_u$, we have by (3.8) that $[h_{1,v}]_b = [k_1]_b$. Therefore $[h_{1,u}]_b = [h_{1,v}]_b$ and Claim 3.7 holds.

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Theorem IX.3.6 (continued 3)

Theorem IX.3.6. Monodromy Theorem.
Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then $[f_1]_b = [g_1]_b$.

Proof (continued). If $u \in U^-$ and let $\delta > 0$ is again as in Claim 3.7 then there is $v \in U$ with $|u - v| < \delta$. By Claim 3.7 we have $[h_{1,u}]_b = [h_{1,v}]_b$. Since $v \in U$ then by the definition of U, $[h_{1,v}]_b = [h_{1,0}]_b$. Therefore $[h_{1,u}]_b = [h_{1,0}]_b$ and so $u \in U$. So U contains all its points of closure and U is closed in $[0, 1]$. The claim now holds as described above. □

Theorem IX.3.6 (continued 3)

Theorem IX.3.6. Monodromy Theorem.
Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then $[f_1]_b = [g_1]_b$.

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Corollary IX.3.9

Corollary IX.3.9. Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Then there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in D$.

Proof. Fix $a \in D$ and let $z \in G$. If γ is a path in G from a to z and $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ is an analytic continuation of (f, D) along γ (which exists since (f, D) admits unrestricted continuation by hypothesis) then let $F(z, \gamma) = f_1(z)$.

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Proof (continued). To show that F is analytic, let $z \in G$ and let γ and $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ be as above. In Exercise IX.3.A it is to be shown that $F(w) = f_1(w)$ for all w in some neighborhood of z (a "simple argument," according to Conway on page 221). So F is analytic on this neighborhood of z. Since z is an arbitrary point in G, then F is analytic on G. □

Corollary IX.3.9 (continued)

Corollary IX.3.9. Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Then there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in D$.

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