Complex Analysis

Chapter IX. Analytic Continuation and Riemann Surfaces

IX.4. Topological Spaces and Neighborhood Systems—Proofs of Theorems

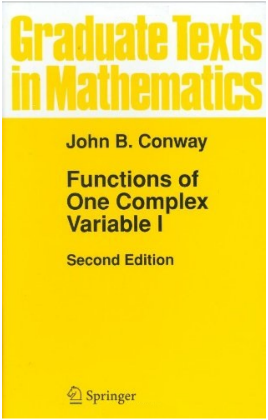

Table of contents

Proposition IX.4.17(b)

Proposition IX.4.17(b).

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x$ in U implies there is a V in $\mathcal{N}_x$ such that $V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

Proof. Let $\{\mathcal{N}_x \mid x \in X\}$ be a neighborhood system on X and let $\mathcal{T}$ be as described. Then $X \in \mathcal{T}$ trivially and $\varnothing \in \mathcal{T}$ vacuously.

Proposition IX.4.17(b)

Proposition IX.4.17(b).

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x \text{ in } U$ implies there is a V in $\mathcal{N}_x$ such that $V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

Proof. Let $\{\mathcal{N}_x \mid x \in X\}$ be a neighborhood system on X and let $\mathcal{T}$ be as described. Then $X \in \mathcal{T}$ trivially and $\varnothing \in \mathcal{T}$ vacuously. Let $U_1, U_2, \ldots, U_n \in \mathcal{T}$ and put $U - \cap_{j=1}^n U_j$. If $x \in U$ then, by definition of $\mathcal{T}$, for each j there is $V_j \in \mathcal{N}_x$ such that $V_j \subset U_j$. From part (b) of the definition of $\mathcal{N}_x$ (Definition IX.4.16) and by mathematical induction, there is $V \in \mathcal{N}_x$ such that $V \subset \cap_{j=1}^n V_j \subset \cap_{j=1}^n U_j = U$. So by the definition of $\mathcal{T}$, $U \in \mathcal{T}$.

Proposition IX.4.17(b)

Proposition IX.4.17(b).

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x \text{ in } U$ implies there is a V in $\mathcal{N}_x$ such that $V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

Proof. Let $\{\mathcal{N}_x \mid x \in X\}$ be a neighborhood system on X and let $\mathcal{T}$ be as described. Then $X \in \mathcal{T}$ trivially and $\varnothing \in \mathcal{T}$ vacuously. Let $U_1, U_2, \ldots, U_n \in \mathcal{T}$ and put $U - \cap_{j=1}^n U_j$. If $x \in U$ then, by definition of $\mathcal{T}$, for each j there is $V_j \in \mathcal{N}_x$ such that $V_j \subset U_j$. From part (b) of the definition of $\mathcal{N}_x$ (Definition IX.4.16) and by mathematical induction, there is $V \in \mathcal{N}_x$ such that $V \subset \cap_{j=1}^n V_j \subset \cap_{j=1}^n U_j = U$. So by the definition of $\mathcal{T}$, $U \in \mathcal{T}$. If $U_i \in \mathcal{T}$ for all $i \in I$ then for a given $s \in \cup_{i \in I} U_i$, there is $i' \in I$ with $x \in U_{i'}$. Since $U_{i'} \in \mathcal{T}$ there is $V \in \mathcal{N}_x$ with $V \subset U_{i'}$ (by the definition of $\mathcal{T}$). Then $V \subset \cup_{i \in I} U_i$ and so $\cup_{i \in I} U_i \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a topology on X.

Proposition IX.4.17(b)

Proposition IX.4.17(b).

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x \text{ in } U$ implies there is a V in $\mathcal{N}_x$ such that $V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

Proof. Let $\{\mathcal{N}_x \mid x \in X\}$ be a neighborhood system on X and let $\mathcal{T}$ be as described. Then $X \in \mathcal{T}$ trivially and $\varnothing \in \mathcal{T}$ vacuously. Let $U_1, U_2, \ldots, U_n \in \mathcal{T}$ and put $U - \cap_{j=1}^n U_j$. If $x \in U$ then, by definition of $\mathcal{T}$, for each j there is $V_j \in \mathcal{N}_x$ such that $V_j \subset U_j$. From part (b) of the definition of $\mathcal{N}_x$ (Definition IX.4.16) and by mathematical induction, there is $V \in \mathcal{N}_x$ such that $V \subset \cap_{j=1}^n V_j \subset \cap_{j=1}^n U_j = U$. So by the definition of $\mathcal{T}$, $U \in \mathcal{T}$. If $U_i \in \mathcal{T}$ for all $i \in I$ then for a given $s \in \cup_{i \in I} U_i$, there is $i' \in I$ with $x \in U_{i'}$. Since $U_{i'} \in \mathcal{T}$ there is $V \in \mathcal{N}_x$ with $V \subset U_{i'}$ (by the definition of $\mathcal{T}$). Then $V \subset \cup_{i \in I} U_i$ and so $\cup_{i \in I} U_i \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a topology on X.

Proposition IX.4.17(b) (continued)

Proposition IX.4.17(b).

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x$ in U implies there is a V in $\mathcal{N}_x$ such that $V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

Proof (continued). Fix $x \in X$ and let $U \in \mathcal{N}_x$. If $y \in U$ then for $V \in \mathcal{N}_y$ we have $y \in U \cap V$ and so by part (c) of the definition of neighborhood system (Definition IX.4.16; we take z in the definition to be y here) there is $W \in \mathcal{N}_y$ such that $W \subset U \cap V \subset U$. So, by the definition of $\mathcal{T}$, $U \in \mathcal{T}$ and hence $\mathcal{N}_x \subset \mathcal{T}$, as claimed. □