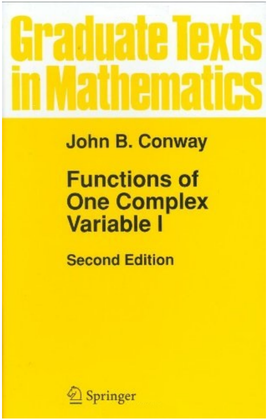
Complex Analysis

Chapter IX. Analytic Continuation and Riemann Surfaces

IX.5. The Sheaf of Germs of Analytic Functions on an Open Set—Proofs of Theorems



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Then $\{\mathcal{N}_{(a,[f]_a)} \mid (a, [f]_a) \in \mathscr{S}(G)\}$ is a neighborhood system on $\mathscr{S}(G)$ and the induced topology is Hausdorff. Furthermore, the induced topology makes the projection map $\rho : \mathscr{S}(G) \to G$ continuous.

Proof. Fix $(z, [f]_a)$ in $\mathscr{S}(G)$. We use Definition IX.4.16 to show that $\{\mathcal{N}_{(a,[f]_a)} \mid (a, [f]_a) \in \mathscr{S}(G)\}$ is a neighborhood system on $\mathscr{S}(G)$ (along with the observation that part (c) of the definition implies part (b) of the definition; see page 226 or the note in the class notes after Definition IX.4.16).

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$$(b, [h]_b) \in N(g_1, B_1) \cap N(g_2, B_2). \qquad (5.4)$$

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Theorem IX.5.3 (continued 1)

Proof (continued). We need to find function element (k, W) such that $N(k, W) \in \mathcal{N}_{(b,[h]_b)}$ and $N(k, W) \subset N(g_1, B_1) \cap N(g_2, B_2)$. If follows from (5.4) and the definition of $N(g, B)$ that $b \in B_1 \sup B_2$ and $[h]_b = [g_1]_b = [g_2]_b$ (as argued above when justifying part(a) of Definition IX.4.16). Since $[h]_b = [g_1]_b = [g_2]_b$ then there is a neighborhood W of b such that $h(z) = g_1(z) = g_2(z)$ for all $z \in W$, and so $W \subset B_1 \cap B_2$. Now $N(h, W) \in \mathcal{N}_{(b,[h]_b)}$.

Theorem IX.5.3 (continued 1)

Proof (continued). We need to find function element (k, W) such that $N(k, W) \in \mathcal{N}_{(b,[h]_b)}$ and $N(k, W) \subset N(g_1, B_1) \cap N(g_2, B_2)$. If follows from (5.4) and the definition of $N(g, B)$ that $b \in B_1 \sup B_2$ and $[h]_b = [g_1]_b = [g_2]_b$ (as argued above when justifying part(a) of Definition IX.4.16). Since $[h]_b = [g_1]_b = [g_2]_b$ then there is a neighborhood W of b such that $h(z) = g_1(z) = g_2(z)$ for all $z \in W$, and so $W \subset B_1 \cap B_2$. Now $N(h, W) \in \mathcal{N}_{(b,[h]_b)}$. Finally, $N(h, W) = \{(z, [h]_z) \mid z \in W\}$ so for any $(z, [h]_z) \in N(h, W)$, we have $(z, [h]_z) \in N(g_1, B_1) \cap N(g_2, B_2)$ since $z \in W$ implies $z \in B_1 \cap B_2$ and h is defined on W. That is, $N(h, W) \subset N(g_1, B_1) \cap N(g_2, B_2)$. So part (c) of Definition IX.4.16 is satisfied and $\{\mathcal{N}_{(a,[f]_a)} \mid (a, [f]_a) \in \mathscr{S}(G)\}$ is a neighborhood system on $\mathscr{S}(G)$.

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To show the topology induced by the neighborhood system is Hausdorff, we use Corollary IX.4.19. Let $(a, [f]_1)$ and $(b, [g]_b)$ be distinct points of $\mathscr{S}(G)$.

Theorem IX.5.3 (continued 1)

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Theorem IX.5.3 (continued 2)

Proof (continued). We must find neighborhood $N(f, A)$ of $(a, [f]_a)$ and a neighborhood $N(g, B)$ of $(b, [g]_b)$ such that $N(f, A) \cap N(g, B) = \varnothing$. Notice that as part of the neighborhood system, both $N(f, A)$ and $(N(g, B)$ are in fact open sets. Notice that $(a, [f]_1) \neq (b, [g]_b)$ implies that either $a \neq b$, or $a = b$ and $[f]_a \neq [g]_a$. If $a \neq b$ then let A and B be disjoint disks about a and b respectively (which can be done since $\mathbb{C}$ is Hausdorff); then $N(f, A) \cap N(g, B) = \varnothing$. If $a = b$ and $[f]_a \neq [g]_a$, then there is a disk $B = B(a; r)$ such that both f and g are defined on D but $f(z) \neq g(z)$ for $0 < |z - a| < r$.

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Theorem IX.5.3 (continued 3)

Proof (continued). ASSUME $N(f, D) \cap N(g, D) \neq \varnothing$ and that $(z, [h]_z) \in N(f, D) \cap N(g, D)$. Then $z \in D$, $[h]_z = [f]_z$, and $[h]_z = [g]_z$. But then f and g are equal on some neighborhood of z, CONTRADICTING the existence of $B(z, ; r)$ above. So the assumption that $N(f, D) \cap N(g, D) \neq \varnothing$ is false and hence neighborhoods $N(f, D)$ and $N(g, D)$ of $[f]_z$ and $[g]_z$ respectively are disjoint. That is, the induced topology is Hausdorff.

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Finally for the continuity of $\rho : \mathscr{S}(G) \to G$, let $U \subset G$ be open. Let $(x, [f]_z) \in \rho^{-1}(U)$. Then $\rho((z, [f]_z)) = z \in U$. Let $D \subset G$ be a disk containing z on which f is defined and such that $D \subset U$ and $N(f, D) \subset \rho^{-1}(U)$. By Exercise IX.4.3, we have that ρ is continuous. □

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Proposition IX.5.8

Proposition IX.5.8. Let G be an open subset of the complex plane and let U be an open connected subset of G such that there is an analytic function f defined on U. Then $N(f, U) = \{(z, [f]_z) \mid z \in U\}$ is arcwise connected in $\mathscr{S}(G)$.

Proof. Let $(a, [f]_a)$ and $(b, [f]_b)$ be any two points in $N(f, U)$. Then $a, b \in U$. Since U is a region, by Theorem II.2.3, there is a (polygonal) path $\gamma : [0, 1] \to U$ from a to b. Define $\sigma : [0, 1] \to N(f, U)$ as $\sigma(t) = (\gamma(t), [f]_{\gamma(t)})$. Then $\sigma(0) = (\gamma(0), [f]_{\gamma(0)}) = (a, [f]_a)$ and $\sigma(1) = (\gamma(1), [f]_{\gamma(1)}) = (b, [f]_b)$. We need to show that σ is continuous.

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Fix $t \in [0, 1]$ and let $N(g, V) = \{(z, [g]_z) \mid z \in V\}$ be a neighborhood of $\sigma(t) = (\gamma(t), [f]_{\gamma(t)})$. Then $\gamma(t) \in V$ and $[f]_{\gamma(t)} = [g]_{\gamma(t)}$. So f and g agree on a neighborhood of $\gamma(t)$; that is, there is $r > 0$ such that $B(\gamma(t); r) \subset U \cap V$ and $f(z) = g(z)$ for all $z \in B(\gamma(t); r)$. Since γ is continuous at t, there is $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - r| < \delta$.

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Proof. Let $(a, [f]_a)$ and $(b, [f]_b)$ be any two points in $N(f, U)$. Then $a, b \in U$. Since U is a region, by Theorem II.2.3, there is a (polygonal) path $\gamma : [0, 1] \to U$ from a to b. Define $\sigma : [0, 1] \to N(f, U)$ as $\sigma(t) = (\gamma(t), [f]_{\gamma(t)})$. Then $\sigma(0) = (\gamma(0), [f]_{\gamma(0)}) = (a, [f]_a)$ and $\sigma(1) = (\gamma(1), [f]_{\gamma(1)}) = (b, [f]_b)$. We need to show that σ is continuous.

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Proposition IX.5.8 (continued)

Proposition IX.5.8. Let G be an open subset of the complex plane and let U be an open connected subset of G such that there is an analytic function f defined on U. Then $N(f, U) = \{(z, [f]_z) \mid z \in U\}$ is arcwise connected in $\mathscr{S}(G)$.

Proof (continued). So $s \in (t - \delta, t + \delta)$ implies that $\gamma(s) \in B(\gamma(t); r)$ and so $f(\gamma(s)) = g(\gamma(s))$ for all $s \in (t - \delta, t + \delta)$. So for $s \in (t - \delta, t + \delta)$, $\sigma(s) = (\gamma(s), [f]_{\gamma(s)}) = (\gamma(s), [g]_{\gamma(s)}) \in N(g, V)$. Since $\sigma((t - \delta, t + \delta)) \subset g, V)$, by Exercise IX.4.3, σ is continuous and the claim follows. □

Corollary IX.5.9

Corollary IX.5.9. The sheaf of germs of analytic functions on G, $\mathscr{S}(G)$, is locally arcwise connected and the components of $\mathscr{S}(G)$ are open arcwise connected sets.

Proof. To show locally arcwise connected, we need to show that for each point $(z, [f]_z) \in \mathscr{S}(G)$ and each open set U_1 containing $(z, [f]_z)$ there is an open arcwise connected set U_2 containing $(z, [f]_z)$ with $U_z \subset U_1$. For point $(z, [f]_z)$ and open set U_1 containing $(z, [f]_a)$, by Theorem IX.5.3(b) there is a neighborhood system element $U_2 = N(f, U) \subset U_1$ (also recall that a neighborhood system is a "basis" for the induced topology).

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Since we know that $\mathscr{S}(G)$ is locally arcwise connected, by Proposition IX.5.6(b), each component of $\mathscr{S}(G)$ is open. Since "components" are by definition connected, by Proposition IX.5.7 the components of $\mathscr{S}(G)$ are arcwise connected sets. □

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Since we know that $\mathscr{S}(G)$ is locally arcwise connected, by Proposition IX.5.6(b), each component of $\mathscr{S}(G)$ is open. Since "components" are by definition connected, by Proposition IX.5.7 the components of $\mathscr{S}(G)$ are arcwise connected sets. □

Theorem IX.5.10

Theorem IX.5.10. There is a path in $\mathscr{S}(G)$ from $(a, [f]_a)$ to $(b, [g]_b)$ if and only if there is a path γ in G from a to b such that $[g]_b$ is an analytic continuation of $[f]_a$ along γ.

Proof. Suppose that $\sigma : [0,1] \to \mathscr{S}(G)$ is a path with $\sigma(0) = (a, [f]_a)$ and $\sigma(1) = (b, [g]_b)$. With $\rho : \mathscr{S}(G) \to \mathbb{C}$ defined as $\rho((z, [f]_z)) = z$ as the projection map we have that $\gamma = \rho \circ \sigma$ is a path in G from $\gamma(0) = \rho(\sigma(0)) = \rho((a, [f]_a)) = a$ to $\sigma(1) = \rho(\sigma(1)) = \rho((b, [g]_b)) = b$ (notice that *rho* is continuous by Theorem IX.5.3, the "furthermore" part).

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Proof. Suppose that $\sigma : [0,1] \to \mathscr{S}(G)$ is a path with $\sigma(0) = (a, [f]_a)$ and $\sigma(1) = (b, [g]_b)$. With $\rho : \mathscr{S}(G) \to \mathbb{C}$ defined as $\rho((z, [f]_z)) = z$ as the projection map we have that $\gamma = \rho \circ \sigma$ is a path in G from $\gamma(0) = \rho(\sigma(0)) = \rho((a, [f]_a)) = a$ to $\sigma(1) = \rho(\sigma(1)) = \rho((b, [g]_b)) = b$ (notice that *rho* is continuous by Theorem IX.5.3, the "furthermore" part). Since $\sigma(t) \in \mathscr{S}(G) = \{(z, [f]_z) \mid z \in G \text{ and } f \text{ is analytic at } z\}$ for each $t \in [0,1]$, then there is a germ $[f_t]_{\gamma(t)} \in \mathscr{S}(G)$ such that $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ for each $t \in [0,1]$. We claim that $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is the required continuation of $[f]_a$ along γ. First, $\sigma(0) = [f_0]_{\gamma(0)} = [f]_a$ and $\sigma(1) = [f_1]_{\gamma(1)} = [g]_b$. For each $t \in [0,1]$, let D_t be a disk about $z = \gamma(t)$ such that $D_t \subset G$ and f_t is analytic on D_t (this can be done by either the definition of sheaf of the germs $\mathscr{S}(G)$ or the definition of germ itself, $[f_t]_{\gamma(t)}$).

Theorem IX.5.10 (continued)

Proof (continued). Fix $t \in [0, 1]$. Now $N(f_t, D_t) = \{(z, [f]_z) \mid z \in D_t\}$ is a neighborhood of $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ (it is an element of the neighborhood system which induces the topology) and σ is continuous so there is $\delta > 0$ such that $\sigma((t - \delta, t + \delta)) \subset N(f_t, D_t)$. That is, if $|s - t| < \delta$ then $\sigma(s) = (\gamma(s), [f_s]_{\gamma(s)}) \in N(f_t, D_t) = \{(s, [f_t]_z) \mid z \in D_t\}$ and so $z = \gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. So by the Definition IX.2.2, $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of (f, D_0) to (g, D_1) along γ.

Theorem IX.5.10 (continued)

Proof (continued). Fix $t \in [0,1]$. Now $N(f_t, D_t) = \{(z, [f]_z) \mid z \in D_t\}$ is a neighborhood of $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ (it is an element of the neighborhood system which induces the topology) and σ is continuous so there is $\delta > 0$ such that $\sigma((t-\delta, t+\delta)) \subset N(f_t, D_t)$. That is, if $|s-t| < \delta$ then $\sigma(s) = (\gamma(s), [f_s]_{\gamma(s)}) \in N(f_t, D_t) = \{(s, [f_t]_z) \mid z \in D_t\}$ and so $z = \gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. So by the Definition IX.2.2, $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of (f, D_0) to (g, D_1) along γ.

Now suppose that γ is a curve in G from a to b and $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of $(f_0, D_0) = (f, D_0)$ to $(f_1, D_1) = (g, D_1)$ along γ. Define $\sigma : [0,1] \to \mathscr{S}(G)$ as $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$. We need to show that σ is a path from $(a, [f]_a)$ to $(b, [g]_b)$.

Theorem IX.5.10 (continued)

Proof (continued). Fix $t \in [0,1]$. Now $N(f_t, D_t) = \{(z, [f]_z) \mid z \in D_t\}$ is a neighborhood of $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ (it is an element of the neighborhood system which induces the topology) and σ is continuous so there is $\delta > 0$ such that $\sigma((t-\delta, t+\delta)) \subset N(f_t, D_t)$. That is, if $|s-t| < \delta$ then $\sigma(s) = (\gamma(s), [f_s]_{\gamma(s)}) \in N(f_t, D_t) = \{(s, [f_t]_z) \mid z \in D_t\}$ and so $z = \gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. So by the Definition IX.2.2, $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of (f, D_0) to (g, D_1) along γ.

Now suppose that γ is a curve in G from a to b and $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of $(f_0, D_0) = (f, D_0)$ to $(f_1, D_1) = (g, D_1)$ along γ. Define $\sigma : [0,1] \to \mathscr{S}(G)$ as $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$. We need to show that σ is a path from $(a, [f]_a)$ to $(b, [g]_b)$. First, $\sigma(0) = (\gamma(0), [f_0]_{\gamma(0)}) = (a, [f]_a)$ and $\sigma(1) = (\gamma(1), [f_1]_{\gamma(1)}) = (b, [g]_b)$. Then $\rho \circ \sigma(t) = \rho(\sigma(t)) = \rho((\gamma(t), [f_t]_{\gamma(t)})) = \gamma(t)$. The argument of the first paragraph of this proof can be used to show that σ is continuous (see Exercise IX.5.B) and the result follows. □

Theorem IX.5.10 (continued)

Proof (continued). Fix $t \in [0,1]$. Now $N(f_t, D_t) = \{(z, [f]_z) \mid z \in D_t\}$ is a neighborhood of $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ (it is an element of the neighborhood system which induces the topology) and σ is continuous so there is $\delta > 0$ such that $\sigma((t-\delta, t+\delta)) \subset N(f_t, D_t)$. That is, if $|s-t| < \delta$ then $\sigma(s) = (\gamma(s), [f_s]_{\gamma(s)}) \in N(f_t, D_t) = \{(s, [f_t]_z) \mid z \in D_t\}$ and so $z = \gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. So by the Definition IX.2.2, $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of (f, D_0) to (g, D_1) along γ.

Now suppose that γ is a curve in G from a to b and $\{[f_t]_{\gamma(t)} \mid 0 \le t \le 1\}$ is an analytic continuation of $(f_0, D_0) = (f, D_0)$ to $(f_1, D_1) = (g, D_1)$ along γ. Define $\sigma : [0,1] \to \mathscr{S}(G)$ as $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$. We need to show that σ is a path from $(a, [f]_a)$ to $(b, [g]_b)$. First,
$\sigma(0) = (\gamma(0), [f_0]_{\gamma(0)}) = (a, [f]_a)$ and $\sigma(1) = (\gamma(1), [f_1]_{\gamma(1)}) = (b, [g]_b)$. Then $\rho \circ \sigma(t) = \rho(\sigma(t)) = \rho((\gamma(t), [f_t]_{\gamma(t)})) = \gamma(t)$. The argument of the first paragraph of this proof can be used to show that σ is continuous (see Exercise IX.5.B) and the result follows. □

Theorem IX.5.11

Theorem IX.5.11. Let $\mathscr{C} \subset \mathscr{S}(G)$ and let $(a, [f]_a) \in \mathscr{C}$. Then $\mathscr{C}$ is a component of $\mathscr{S}(G)$ if and only if

$$\mathscr{C} = \{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\}.$$

Proof. Suppose is a component of $\mathscr{S}(G)$. By Corollary IX.5.9, $\mathscr{C}$ is an open arcwise connected subset of $\mathscr{S}(G)$. That is, for each $(b, [g]_b) \in \mathscr{C}$ there is a path from $(a, [f]_a)$ to $(b, [g]_b)$. So by Theorem IX.5.10 there is a path γ from a to b such that $[g]_b$ is an analytic continuation of $[f]_a$ along γ.

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Proof. Suppose is a component of $\mathscr{S}(G)$. By Corollary IX.5.9, $\mathscr{C}$ is an open arcwise connected subset of $\mathscr{S}(G)$. That is, for each $(b, [g]_b) \in \mathscr{C}$ there is a path from $(a, [f]_a)$ to $(b, [g]_b)$. So by Theorem IX.5.10 there is a path γ from a to b such that $[g]_b$ is an analytic continuation of $[f]_a$ along γ. That is,

$$\mathscr{C} \subset \{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\}.$$

Conversely, if $[g]_b$ is the analytic continuation of $[f]_a$ along some curve γ in G (technically, by Definition IX.2.2, (g, D_1) is the analytic continuation of (f, D_0)), then $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ is a path in $\mathscr{S}(G)$ from $(a, [f]_a)$ to $(b, [g]_b)$ and so the set $\mathscr{C}$ of all such $[g]_b$ is arcwise connected in $\mathscr{S}(G)$ and by Proposition IX.5.6 is connected.

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Conversely, if $[g]_b$ is the analytic continuation of $[f]_a$ along some curve γ in G (technically, by Definition IX.2.2, (g, D_1) is the analytic continuation of (f, D_0)), then $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ is a path in $\mathscr{S}(G)$ from $(a, [f]_a)$ to $(b, [g]_b)$ and so the set $\mathscr{C}$ of all such $[g]_b$ is arcwise connected in $\mathscr{S}(G)$ and by Proposition IX.5.6 is connected.

Theorem IX.5.11 (continued)

Proof (continued). So $[g]_b$ is in the component of $\mathscr{S}(G)$ which contains $(a, [f]_a)$. That is,
$\{(b, [g]_b) \mid [g]_b$ is the continuation of $[f]_a$ along some curve in $G\} \supset \mathbb{C}$.
So the sets are equal, as claimed.

Now suppose that $\mathscr{C}$ consists of all points $(b, [g]_b)$ such that $[g]_b$ is an analytic continuation of $[f]_a$. Then there is a path γ from a to b along which the analytic continuation occurs. Then $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ is a path (arc) from $(a, [f]_1)$ to $(b, [g]_b)$ and so $\mathscr{C}$ is arcwise connected and so by Proposition IX.5.6 is connected. So $\mathscr{C}$ is a subset of the components of $\mathscr{S}(G)$ containing $[f]_a$.

Theorem IX.5.11 (continued)

Proof (continued). So $[g]_b$ is in the component of $\mathscr{S}(G)$ which contains $(a, [f]_a)$. That is,
$\{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\} \supset \mathbb{C}$.
So the sets are equal, as claimed.

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Theorem IX.5.11 (continued)

Proof (continued). So $[g]_b$ is in the component of $\mathscr{S}(G)$ which contains $(a, [f]_a)$. That is,
$\{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\} \supset \mathbb{C}$.
So the sets are equal, as claimed.

Now suppose that $\mathscr{C}$ consists of all points $(b, [g]_b)$ such that $[g]_b$ is an analytic continuation of $[f]_a$. Then there is a path γ from a to b along which the analytic continuation occurs. Then $\sigma(t) = (\gamma(t), [f_t]_{\gamma(t)})$ is a path (arc) from $(a, [f]_1)$ to $(b, [g]_b)$ and so $\mathscr{C}$ is arcwise connected and so by Proposition IX.5.6 is connected. So $\mathscr{C}$ is a subset of the components of $\mathscr{S}(G)$ containing $[f]_a$. If $\mathscr{C}_1$ is the component of $\mathscr{S}(G)$ containing $\mathscr{C}_1 \subset \mathscr{C}$ (where $\mathscr{C}$ in the first paragraph plays the role in $\mathscr{C}_1$ here and $\{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\}$ in the first paragraph plays the role of $\mathscr{C}$ here). Therefore $\mathscr{C} = \mathscr{C}_1$ and the claim holds. □

Theorem IX.5.15

Theorem IX.5.15. Let $\mathscr{F}$ be a complete analytic function with base space G and let $(\mathscr{R}, \rho)$ be its Riemann surface. Then $\rho : \mathscr{R} \to G$ is an open continuous map. Also, if $(a, [f]_a)$ is a point in $\mathscr{R}$ then there is a neighborhood $N(f, D)$ of $(a, [f]_a)$ such that ρ maps $N(f, D)$ homeomorphically onto an open disk in $\mathbb{C}$.

Proof. As commented above, $\mathscr{R}$ is a component of $\mathscr{S}(\mathbb{C})$. Since $\rho : \mathscr{S}(\mathbb{C}) \to \mathbb{C}$ is continuous by Theorem IX.5.3, then the restriction of ρ to $\mathscr{R} \subset \mathscr{S}(\mathbb{C})$ is continuous and the restriction maps $\mathscr{R} \to G$.

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Proof. As commented above, $\mathscr{R}$ is a component of $\mathscr{S}(\mathbb{C})$. Since $\rho : \mathscr{S}(\mathbb{C}) \to \mathbb{C}$ is continuous by Theorem IX.5.3, then the restriction of ρ to $\mathscr{R} \subset \mathscr{S}(\mathbb{C})$ is continuous and the restriction maps $\mathscr{R} \to G$. To show that $\rho : \mathscr{R} \to G$ is an open map, it suffices to show that $\rho(N(f, U))$ is open for neighborhood element (or "basis element" of the topology) of arbitrary function element (f, U) is open, by Exercise IX.4.4. But $\rho(N(f, U)) = U$ and U is open by the definition of function element (U is a region). Therefore $\rho : \mathscr{R} \to G$ is a continuous open map.

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Theorem IX.5.15 (continued)

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Proof. For $(a, [f]_a) \in \mathscr{R}$, let D be an open disk such that $(f, D) \in [f]_a$. Then $\rho : N(f, D) \to D$ is an open, continuous map by the previous paragraph and is onto. To show that ρ is a homeomorphism we only need now to show that it is one to one on $N(f, D)$. But if $(b, [f]_b)$ and $(c, [f]_c)$ are distinct points of $N(f, D)$ then $b \neq c$ and $\rho((b, [f]_b)) = b \neq c = \rho((c, [f]_c))$. So ρ is also one to one and hence $\rho : N(f, D) \to D$ is a homeomorphism, as claimed. □

Theorem IX.5.15 (continued)

Theorem IX.5.15. Let $\mathscr{F}$ be a complete analytic function with base space G and let $(\mathscr{R}, \rho)$ be its Riemann surface. Then $\rho : \mathscr{R} \to G$ is an open continuous map. Also, if $(a, [f]_a)$ is a point in $\mathscr{R}$ then there is a neighborhood $N(f, D)$ of $(a, [f]_a)$ such that ρ maps $N(f, D)$ homeomorphically onto an open disk in $\mathbb{C}$.

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