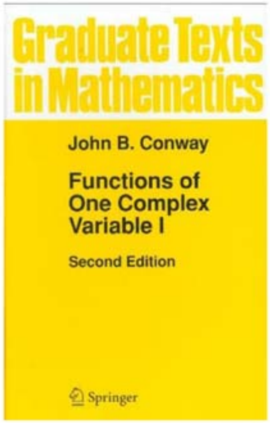
Complex Analysis

Chapter IX. Analytic Continuation and Riemann Surfaces
IX.6. Analytic Manifolds—Proofs of Theorems



Proposition IX.6.3

Proposition IX.6.3. Let (X, Φ) be an analytic surface.

(a) Let V be an open connected subset of X. If

$$\Phi_V = \{(U \cap V, \varphi) \mid (U, \varphi) \in \Phi\}$$

then (V, Φ_V) is an analytic surface.

(b) If Ω is a topological space such that there is a homeomorphism h of X onto Ω then with

$$\Phi = \{(h(U), \varphi \circ h^{-1}) \mid (U, \varphi) \in \Phi\}$$

we have that (Ω, Ψ) is an analytic surface.

Proof. (a) First, since every element of X is in at least one member of Φ then every element of V is in at least one element of Φ_V and part (i) of the definition is satisfied. Second, if $(V_a, \varphi_a), (V_b, \varphi_b) \in \Phi_V$ with $V_a \cap V_b \neq \varnothing$ then $\varphi_a \circ \varphi_b^{-1}$ is an analytic function of $\varphi_b(U_a \cap U_b)$ onto $\varphi_a(U_a \cap U_b)$.

Theorem IX.6.3 (continued 1)

Proof (continued). Since $\varphi_b(V_a \cap V_b) \subset \varphi_b(U_a \cap U_b)$ then $\varphi_a \circ \varphi_b^{-1}$ is also analytic on $\varphi_b(V_a \cap V_b)$ and

$$\begin{aligned}\varphi_a \circ \varphi_b^{-1}(\varphi_a(V_a \cap V_b)) &= \varphi_a \circ (\varphi_b^{-1}\varphi_b)(V_a \cap V_b) \\ &\quad \text{since function composition is associative} \\ &= \varphi_a(V_a \cap V_b)\end{aligned}$$

and so $\varphi_a \circ \varphi_b^{-1}$ maps $\varphi_b(V_a \cap V_b)$ onto $\varphi_a(V_a \cap V_b)$. So part (ii) of the definition is satisfied.

(b) Since X is connected and X is homeomorphic to Ω then Ω is connected. Since $h : X \to \Omega$ is a homeomorphism then for any $(U, \varphi) \in \Phi$ we have that $\varphi \circ h^{-1}$ is a homeomorphism from Ω from Ω to $\mathbb{C}$ which maps an open set $h(U)$ to an open set $\varphi \circ h^{-1}(h(U)) = \varphi(U)$ of the plane $\mathbb{C}$. So each $(h(U), \varphi \circ h^{-1}) \in \Psi$ is a coordinate patch. Next, since $h : X \to \Omega$ is onto, each point in Ω is contained in at least one member of Ψ.

Theorem IX.6.3 (continued 2)

Proof (continued). Now we check part (ii) of the definition of analytic surface. Let $(U, \varphi), (V, \mu) \in \Phi$ (so that $(h(U), \varphi \circ h^{-1}), (h(V), \mu \circ h^{-1}) \in \Psi$) such that $h(U) \cap h(V) \neq \varnothing$. But then

$$h^{-1}(h(U) \cap h(V)) - h^{-1}(h(U)) \cap h^{-1}(h(V)) = U \cap V \neq \varnothing.$$

So

$$(\varphi \circ h^{-1}) \circ (\mu \circ h^{-1})^{-1} = (\varphi \circ h^{-1}) \circ (h \circ \mu^{-1}) = \varphi \circ (h \circ h^{-1}) \circ \mu^{-1} = \varphi \circ \mu^{-1}$$

where $\varphi \circ \mu^{-1}$ is analytic since Φ satisfies part (ii) of the definition. Therefore Ψ satisfies part (ii) of the definition and (Ω, Ψ) is an analytic surface, as claimed. □

Proposition IX.6.6

Proposition IX.6.6. Let G be a region in the plane and let f be an analytic function on g with non-vanishing derivative. For $\alpha = (a, f(a)) \in \Gamma = \{(z, f(z)) \mid z \in G\}$. Let D_z be a disk about a such that $D_a \subset G$ and f is one to one on D_a (which is possible since $f'(a) \neq 0$). Let $U_\alpha = \{(z, f(z)) \mid z \in D_a\}$ and define $\varphi_\alpha : U_\alpha \to \mathbb{C}$ by $\varphi_\alpha(z, f(z)) = f(z)$ for each $(z, f(z)) \in U_\alpha$. If Γ is the graph of f and $\Phi = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in \Gamma\}$ then (Γ, Φ) is an analytic surface.

Proof. Since Γ is homeomorphic to G (though the projection homeomorphism p, as described above) and G is connected, then Γ is connected. Fix $\alpha = (a, f(a)) \in \Gamma$. Just as the projection p of $\Gamma \to G$ is a homeomorphism, the projection φ_α is a homeomorphism of U_α onto $f(D_\alpha)$ (except that φ_α projects into the range of f instead of the domain of f). Suppose that $\beta = (b, f(b)) \in \Gamma$ with $U_\alpha \cap U_\beta \neq \varnothing$. We can now show that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is analytic.

Proposition IX.6.6 (continued)

Proposition IX.6.6. Let G be a region in the plane and let f be an analytic function on g with non-vanishing derivative. For $\alpha = (a, f(a)) \in \Gamma = \{(z, f(z)) \mid z \in G\}$. Let D_z be a disk about a such that $D_a \subset G$ and f is one to one on D_a (which is possible since $f'(a) \neq 0$). Let $U_\alpha = \{(z, f(z)) \mid z \in D_a\}$ and define $\varphi_\alpha : U_\alpha \to \mathbb{C}$ by $\varphi_\alpha(z, f(z)) = f(z)$ for each $(z, f(z)) \in U_\alpha$. If Γ is the graph of f and $\Phi = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in \Gamma\}$ then (Γ, Φ) is an analytic surface.

Proof (continued). Since $f : D_b \to \mathbb{C}$ is one to one and analytic, then it has a local inverse $g : \Omega = d(D_b) \to D_b$ such that $f(g(\omega)) = \omega$ for all $\omega \in \Omega$ (by Corollary IV.7.6). Since $\varphi_\beta(U_\beta) = \Omega = f(D_b)$ then $\varphi_\beta^{-1}(\omega) = (g(\omega, \omega)$. Thus $\varphi_\alpha \circ \varphi_\beta^{-1}(\omega) = \varphi_\alpha((g(\omega), \omega)) = \omega$ for each $\omega \in \varphi_\beta(U_\alpha \cap U_\beta)$. So $\varphi_\alpha \circ \varphi_\beta^{-1}$ is constant and hence analytic on $\varphi_\beta(U_\alpha \cap U_\beta)$. That is, (Γ, Φ) is an analytic surface. □