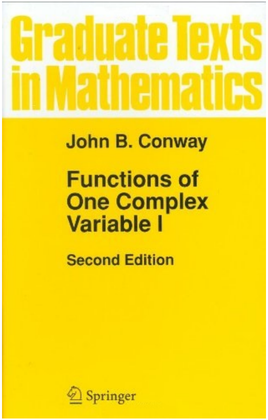
Complex Analysis

Chapter IX. Analytic Continuation and Riemann Surfaces

IX.6. Analytic Manifolds—Proofs of Theorems



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(a) Let V be an open connected subset of X. If

$$\Phi_V = \{(U \cap V, \varphi) \mid (U, \varphi) \in \Phi\}$$

then (V, Φ_V) is an analytic surface.

(b) If Ω is a topological space such that there is a homeomorphism h of X onto Ω then with

$$\Phi = \{(h(U), \varphi \circ h^{-1}) \mid (U, \varphi) \in \Phi\}$$

we have that (Ω, Ψ) is an analytic surface.

Proof. (a) First, since every element of X is in at least one member of Φ then every element of V is in at least one element of Φ_V and part (i) of the definition is satisfied.

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Proof. (a) First, since every element of X is in at least one member of Φ then every element of V is in at least one element of Φ_V and part (i) of the definition is satisfied. Second, if $(V_a, \varphi_a), (V_b, \varphi_b) \in \Phi_V$ with $V_a \cap V_b \neq \varnothing$ then $\varphi_a \circ \varphi_b^{-1}$ is an analytic function of $\varphi_b(U_a \cap U_b)$ onto $\varphi_a(U_a \cap U_b)$.

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Theorem IX.6.3 (continued 1)

Proof (continued). Since $\varphi_b(V_a \cap V_b) \subset \varphi_b(U_a \cap U_b)$ then $\varphi_a \circ \varphi_b^{-1}$ is also analytic on $\varphi_b(V_a \cap V_b)$ and

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and so $\varphi_a \circ \varphi_b^{-1}$ maps $\varphi_b(V_a \cap V_b)$ onto $\varphi_a(V_a \cap V_b)$. So part (ii) of the definition is satisfied.

(b) Since X is connected and X is homeomorphic to Ω then Ω is connected.

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Proof (continued). Now we check part (ii) of the definition of analytic surface. Let $(U, \varphi), (V, \mu) \in \Phi$ (so that $(h(U), \varphi \circ h^{-1}), (h(V), \mu \circ h^{-1}) \in \Psi$) such that $h(U) \cap h(V) \neq \varnothing$. But then

$$h^{-1}(h(U) \cap h(V)) - h^{-1}(h(U)) \cap h^{-1}(h(V)) = U \cap V \neq \varnothing.$$

So

$$(\varphi \circ h^{-1}) \circ (\mu \circ h^{-1})^{-1} = (\varphi \circ h^{-1}) \circ (h \circ \mu^{-1}) = \varphi \circ (h \circ h^{-1}) \circ \mu^{-1} = \varphi \circ \mu^{-1}$$

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Proof. Since Γ is homeomorphic to G (though the projection homeomorphism p, as described above) and G is connected, then Γ is connected. Fix $\alpha = (a, f(a)) \in \Gamma$. Just as the projection p of $\Gamma \to G$ is a homeomorphism, the projection φ_α is a homeomorphism of U_α onto $f(D_\alpha)$ (except that φ_α projects into the range of f instead of the domain of f).

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