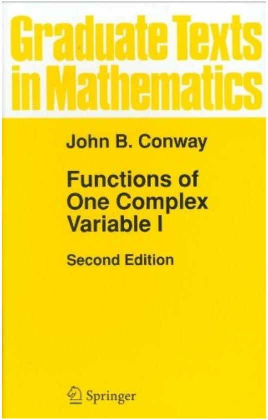
Complex Analysis

Chapter V. Singularities

V.1. Classification of Singularities—Proofs of Theorems



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Proof. Suppose f is analytic in $\{z \mid 0 < |z - a| < R\}$ and define $g(z) = \begin{cases} (z-a)f(z) & \text{for } z \neq a \\ 0 & \text{for } z = a. \end{cases}$ Suppose $\lim_{z \to a}(z - a)f(z) = 0$; then g is continuous on $B(z; R)$. We now show that g is analytic and hence that a is a removable singularity of f.

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We show that g is analytic by applying Morera's Theorem. Let T be a triangle in $B(a; R)$ and let Δ be the inside of T together with T. (1) If $a \notin \Delta$ then $T \sim 0$ in $\{z \mid 0 < |z-a| < R\}$ and so by Cauchy's Theorem (Theorem IV.6.6), $\int_T g = 0$.

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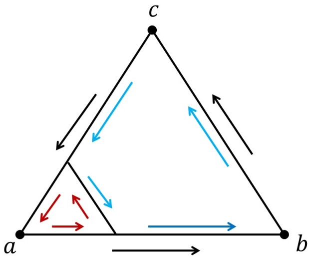
Theorem V.1.2 (continued 1)

Proof (continued). (2) If a is vertex of T then we have $T = [a, b, c, a]$. Let $x \in [a, b]$ and $y \in [c, a]$ and form triangle $T_1 = [a, x, y, a]$. If P is a polygon $[x, b, c, y, x]$ then

$$\int_T g(z)\,dz = \int_{T_1} g(z)\,dz + \int_P g(z)\,dz = \int_{T_1} g(z)\,dz$$

since $P \sim 0$ in $\{a \mid 0 < |z - a| < R\}$:



Theorem V.1.2 (continued 3)

Proof (continued). Since g is continuous and $g(a) = 0$, then for any $\varepsilon > 0$, x and y can be chosen such that $|g(z)| < \varepsilon/\ell(T)$ for any z on T_1. Hence $\left|\int_T g(z)\,dz\right| = \left|\int_{T_1} g(z)\,dz\right| < \varepsilon$. Therefore, $\int_T g(z)\,dz = 0$.

(3) If $a \in \Delta$ and $T = [x, y, z, x]$ then consider triangles $T_1 = [x, y, a, x]$, $T_2 = [y, z, a, y]$, and $T_3 = [z, x, a, z]$:

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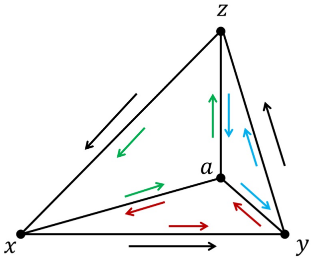

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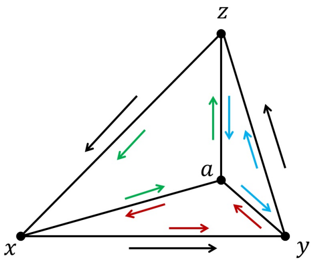

Theorem V.1.2 (continued 4)

Theorem V.1.2. If f has an isolated singularity at a then $z = a$ is a removable singularity if and only if $\lim_{z \to a} (z - a) f(z) = 0$.

Proof (continued). As above, $\int_{T_j} g(z)\,dz = 0$ for $j = 1, 2, 3$ and so $\int_T g(z)\,dz = \int_{T_1} g(z)\,dz + \int_{T_2} g(z)\,dz + \int_{T_2} g(z)\,dz = 0$. Therefore, by (1), (2), and (3), g must be analytic by Morera's Theorem, and f has a removable singularity at $z = a$

Next, suppose f has an isolated, removable singularity at $z = a$. Then there is analytic $g : B(a; R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$ (by definition of isolated singularity). Then $\lim_{z \to a}(z - a) f(z) = \lim_{z \to a}(z - a) g(z) = 0$. □

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Proposition V.1.4

Proposition V.1.4. If G is a region with $a \in G$, and if f is analytic in $G \setminus \{a\}$ with a pole at $z = a$, then there is a positive integer m and an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \dfrac{g(z)}{(z-a)^m}$.

Proof. Suppose f has a pole at $z = a$. Then $\lim_{z\to a} |f(z)| = \infty$ and $\lim_{z\to a} 1/f(z) = 0$. So $1/f$ has a removable discontinuity at $z = a$ by Theorem V.1.2. So the function $h(z) = \begin{cases} 1/f(z) & \text{for } z \neq a \\ 0 & \text{for } z = a \end{cases}$ is analytic in $B(a; R)$ for some $R > 0$ (since a pole is an isolated singularity).

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Corollary V.1.18

Corollary V.1.18. Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n(a - z)^n$ be its Laurent expansion in $\text{ann}(a; 0, R)$. Then

(a) $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$,

(b) $a = z$ is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$, and

(c) $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative integers n.

Proof. (a) If $a_n = 0$ for $n \leq -1$ then $g(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ on $B(a; R)$ is analytic and equals f on $\text{ann}(a; 0, R)$. Conversely, if $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ on $\text{ann}(a; 0, R)$, then

$$\lim_{z \to a}(z-a)f(z) = \lim_{z \to a}\left(\sum_{n=0}^{\infty} a_n(z-a)^{n+1}\right) = 0$$

and f has a removable singularity at $z = a$ by Theorem V.1.2.

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Corollary V.1.18 (continued)

Proof (continued). (b) Suppose $a_n = 0$ for $n \leq -(m+1)$. Then $(z-a)^m f(z)$ has a Laurent expansion which has no negative powers of $(z-a)$. By part (a), $(z-a)^m f(z)$ has a removable singularity at $z = a$. So, by definition, f has a pole of order m at $z = a$. Each of these steps is "if and only if," so the converse holds.

(c) For an essential singularity at $z = a$, neither (a) nor (b) holds and hence (c) must be the case (and conversely since both (a) and (b) are "if and only if"). □

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Theorem V.1.21

Theorem V.1.21. Casorati-Weierstrass Theorem.
If f has an essential singularity at $z = a$ then for every $\delta > 0$, $\{f(\text{ann}(a; 0, \delta)\}^- = \mathbb{C}$.

Proof. Suppose that f is analytic in $\text{ann}(a; 0, R)$. We need to show that every complex number is a limit point of $f(\text{ann}(a; 0, \delta))$ (for every δ). If this is NOT the case (we go for a contradiction) then there is $\delta > 0$, $c \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - c| \geq \varepsilon$ for all $z \in G = \text{ann}(a; 0, \delta)$.

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$$\lim_{z \to a} \left| \frac{f(z) - c}{z - a} \right| \geq \lim_{z \to a} \frac{\varepsilon}{|z - a|} = \infty,$$

which implies that $(f(z) - c)/(z - a)$ has a pole at $z = a$. If m is the order of the pole (known to exist by Proposition V.1.4), then $\lim_{z \to a} |z - a|^{m+1} |f(z) - c| = 0$ by Corollary V.1.18(b).

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Theorem V.1.21 (continued)

Proof. We know

$$|z-a|^{m+1}|f(z)-c| \geq |z-a|^{m+1}|f(z)| - |z-a|^{m+1}|c|,$$

or

$$|z-a|^{m+1}|f(z)| \leq |z-a|^{m+1}|f(z)-c| + |z-a|^{m+1}|c|.$$

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and $\lim_{z\to a}|z-a|^{m+1}|f(z)| = 0$ (remember $m \geq 1$). But then Theorem V.1.2 implies that $f(z)(z-a)^m$ has a removable singularity at $z=a$. By Proposition V.1.4, f has a pole of order m, CONTRADICTING the hypothesis that f has an essential singularity at a. □

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