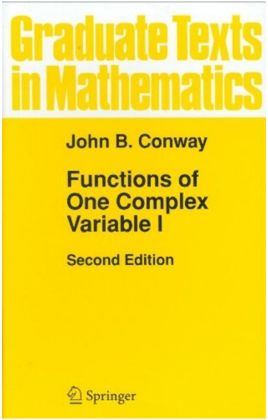
Complex Analysis

Chapter V. Singularities

V.2. Residues—Proofs of Theorems



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Let f be analytic in the region G, except for the isolated singularities $a_1, a_2, \ldots a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G then $\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^m n(\gamma; a_k)\,\mathrm{Res}(f; a_k)$.

Proof. Define $m_k = n(\gamma; a_k)$ for $1 \le k \le m$. Choose positive $r_1, r_2, \ldots, r_m$ such that the disks $\overline{B}(a_k; r_k)$ are disjoint, none of them intersect $\{\gamma\}$, and each disk is contained in G. This can be done since $\{\gamma\}$ is compact (by Theorem II.5.17) and G is open.

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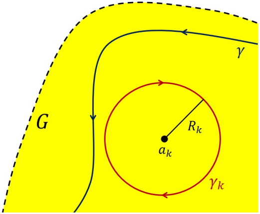

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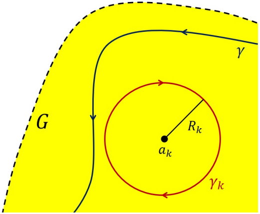

Theorem V.2.2 (continued 1)

Proof (continued). Then for $1 \le j \le m$, $n(\gamma; a_j) + \sum_{k=1}^m n(\gamma_k; a_j) = 0$ (since the disks are disjoint, in fact the terms in the summation are 0 except for when $j = k$ in which case $n(\gamma_k; a_k) = -m_k$). Since $\gamma \approx 0$ in G then, by definition, $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus G$, and since $\overline{B}(a_k; r_k) \subset G$ then $n(\gamma_k; a) = 0$ for all $a \in \mathbb{C} \setminus G$ and for $1 \le k \le m$. So $n(\gamma; a) + \sum_{k=1}^m n(\gamma_k; a) = 0$ for all a not in $G \setminus \{a_1, a_2, \ldots, a_m\}$ (that is, for all $a \in \mathbb{C} \setminus (G \setminus \{a_1, a_2, \ldots, a_m\})$). Since f is analytic in $G \setminus \{a_1, a_2, \ldots, a_m\}$, then by the First Version of Cauchy's Theorem (Theorem IV.5.7) implies

$$0 = \int_\gamma f(z)\,dz + \sum_{k=1}^m \int_{\gamma_k} f(z)\,dz. \qquad (2.3)$$

If $f(z) = \sum_{n=-\infty}^{\infty} b_n (z - a_k)^n$ is the Laurent expansion of f about $z = a_k$, then this series converges uniformly on $\partial B(a_k; r_k)$ by Theorem V.1.11 (Laurent Series Development).

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Theorem V.2.2 (continued 2)

Proof (continued). So the uniform convergence gives

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} \left(\sum_{n=-\infty}^{\infty} b_n (z-a_k)^n \right) = \sum_{-\infty}^{\infty} b_n \left(\int_{\gamma_k} (z-a_k)^n\,dz \right).$$

Now for $n \neq -1$, $(z-a_k)^n$ has a primitive and $\int_{\gamma_k} (z-a_k)^n\,dz = 0$. When $n = -1$,

$$\begin{aligned} b_{-1} \int_{\gamma_k} (z-a_k)^{-1}\,dz &= \operatorname{Res}(f; a_k) \int_{\gamma_k} (z-a_k)^{-1}\,dz \\ &\qquad \text{by the definition of residue} \\ &= \operatorname{Res}(f; a_k) 2\pi i n(\gamma_k; a_k) \\ &\qquad \text{by the definition of winding number.} \end{aligned}$$

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Proof (continued). So (2.3) gives that

$$\int_\gamma f(z)\,dz = -\sum_{k=1}^m \left(\int_{\gamma_k} f(z)\,dz\right)$$

$$= -\sum_{k=1}^m 2\pi i n(\gamma_k; a_k)\mathrm{Res}(f; a_k) = 2\pi i \sum_{k=1}^m n(\gamma; a_k)\mathrm{Res}(f; a_k)$$

since $n(\gamma_k; a_k) = -n(\gamma; a_k)$. Therefore,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{k=1}^m n(\gamma; a_k)\mathrm{Res}(f; a_k). \quad \square$$

Proposition V.2.4

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$$\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Proof. By Proposition V.1.4 and the definition of "pole of order m," we have that $g(z)$ has a removable singularity at $z = a$ and $g(a) = b_0 \neq 0$ (here, we technically mean that $\lim_{z \to a} g(z) = b_0 \neq 0$). Let $g(z) = \sum_{k=1}^{\infty} b_k (z - a)^k$ be the power series of g about $z = a$.

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So this is the Laurent series of f about $z = a$, and so $\text{Res}(f;a) = b_{m-1}$. Since b_{m-1} is the coefficient for $(z-a)^{m-1}$ is the power series representation of g, so $b_{m-1} = g^{(m-1)}(a)/(m-1)!$. □

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