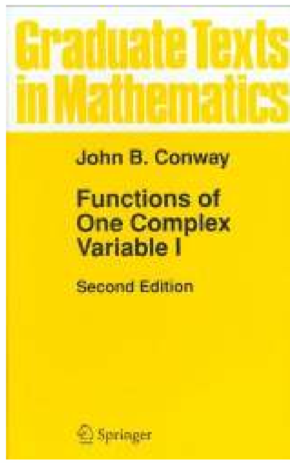
Complex Analysis

Chapter V. Singularities

V.3. The Argument Principle—Proofs of Theorems



Theorem V.3.4

Theorem V.3.4. Argument Principle.
Let f be meromorphic in G with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ repeated according to multiplicity. If γ is a closed rectifiable curve in G where $\gamma \approx 0$ and not passing through $p_1, p_2, \ldots, p_m, z_a, z_2, \ldots, z_n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^n n(\gamma; z_k) - \sum_{j=1}^m n(\gamma; p_j).$$

Proof. By repeated application of (3.1) and (3.2) (applying to each zero and each pole) we have

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^n \frac{1}{z-z_k} - \sum_{j=1}^m \frac{1}{z-p_j} + \frac{g'(z)}{g(z)}$$

where g is analytic on G and nonzero on G.

Theorem V.3.4 (continued)

Proof (continued). Therefore g'/g is analytic and by Cauchy's Theorem (First Version; Theorem IV.5.7), $\int_\gamma g'(z)/g(z)\,dz = 0$. By the definition of winding number, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=1}^n \frac{1}{z-z_k} - \sum_{j=1}^m \frac{1}{z-p_j}\right) dz$$

$$= \sum_{k=1}^n n(\gamma; z_k) - \sum_{j=1}^m n(\gamma; p_j).$$

□

Proposition V.3.7

Proposition V.3.7. Let f be analytic on an open set containing $\overline{B}(a; R)$ and suppose that f is one to one on $B(a; R)$. If $\Omega = f[B(a; R)]$ and γ is the circle $|z-a| = R$, then $f^{-1}(\omega)$ is defined for each $\omega \in \Omega$ by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-\omega}\,dz.$$

Proof. Let $\omega \in \Omega = f(B(a; R))$. Since f is one to one on $B(a; R)$, then the function $f(z) - \omega$ is one to one and so has only one zero in $B(a; R)$ (namely, the element of $B(a; R)$ which is mapped to ω, denoted $f^{-1}(\omega)$). Take $g(z) = z$ and then Theorem V.3.6 gives

$$g(f^{-1}(\omega))n(\gamma; f^{-1}(\omega)) = \frac{1}{2\pi i}\int_\gamma g(z)\frac{(f(z)-\omega)'}{f(z)-\omega}\,dz$$

or $f^{-1}(\omega) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{zf'(z)}{f(z)-\omega}\,dz.$ □

Theorem V.3.8

Theorem V.3.8. Rouche's Theorem.
Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$. Suppose Z_f and Z_g are the number of zeros inside γ, and P_f and P_g are the number of poles inside γ (counted according to their multiplicities) and that $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ. Then $Z_f - P_f = Z_g - P_g$.

Proof. By hypothesis, $\left|\frac{f(z)}{g(z)} + 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$ on γ (g has no zeros on γ, by hypothesis). If $\lambda = f(z)/g(z)$ for some given $z \in \{\gamma\}$ and λ is a nonnegative real number, then the inequality becomes $\lambda + 1 < \lambda + 1$, a contradiction. So meromorphic function $f/g : \{\gamma\} \to \Omega = \mathbb{C} \setminus [0, \infty)$. So there is a branch of the logarithm defined on Ω, say $\mathrm{LOG}(z)$. Then $\mathrm{LOG}(f/g)$ is a primitive of $(f/g)'/(f/g)$ valid on a neighborhood of γ.

Theorem V.3.8 (continued)

Theorem V.3.8. Rouche's Theorem.
Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$. Suppose Z_f and Z_g are the number of zeros inside γ, and P_f and P_g are the number of poles inside γ (counted according to their multiplicities) and that $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ. Then $Z_f - P_f = Z_g - P_g$.

Proof (continued). So

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{f/g} \text{ by Corollary IV.1.22} \\
&= \frac{1}{2\pi i}\int_\gamma \frac{g}{f}\frac{f'g - fg'}{g^2} = \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} \\
&= (Z_f - P_f) - (Z_g - P_g) \text{ by the Argument Principle.} \quad \square
\end{aligned}$$