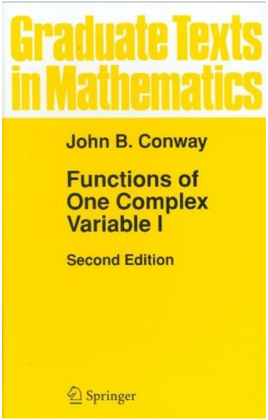
# Complex Analysis

## Chapter V. Singularities

V.3. The Argument Principle—Proofs of Theorems



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# Theorem V.3.4

**Theorem V.3.4. Argument Principle.**
Let $f$ be meromorphic in $G$ with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ repeated according to multiplicity. If $\gamma$ is a closed rectifiable curve in $G$ where $\gamma \approx 0$ and not passing through $p_1, p_2, \ldots, p_m, z_a, z_2, \ldots, z_n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

**Proof.** By repeated application of (3.1) and (3.2) (applying to each zero and each pole) we have

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n}\frac{1}{z-z_k} - \sum_{j=1}^{m}\frac{1}{z-p_j} + \frac{g'(z)}{g(z)}$$

where $g$ is analytic on $G$ and nonzero on $G$.

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# Theorem V.3.4 (continued)

**Proof (continued).** Therefore $g'/g$ is analytic and by Cauchy's Theorem (First Version; Theorem IV.5.7), $\int_\gamma g'(z)/g(z)\,dz = 0$. By the definition of winding number, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=1}^{n}\frac{1}{z-z_k} - \sum_{j=1}^{m}\frac{1}{z-p_j}\right)\,dz$$

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# Proposition V.3.7

**Proposition V.3.7.** Let $f$ be analytic on an open set containing $\overline{B}(a;R)$ and suppose that $f$ is one to one on $B(a;R)$. If $\Omega = f[B(a;R)]$ and $\gamma$ is the circle $|z-a| = R$, then $f^{-1}(\omega)$ is defined for each $\omega \in \Omega$ by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-\omega}\,dz.$$

**Proof.** Let $\omega \in \Omega = f(B(a;R))$. Since $f$ is one to one on $B(a;R)$, then the function $f(z) - \omega$ is one to one and so has only one zero in $B(a;R)$ (namely, the element of $B(a;R)$ which is mapped to $\omega$, denoted $f^{-1}(\omega)$).

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$$g(f^{-1}(\omega))n(\gamma;f^{-1}(\omega) = \frac{1}{2\pi i}\int_\gamma g(z)\frac{(f(z)-\omega)'}{f(z)-\omega}\,dz$$

or $f^{-1}(\omega) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{zf'(z)}{f(z)-\omega}\,dz.$ □

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# Theorem V.3.8

**Theorem V.3.8. Rouche's Theorem.**
Suppose $f$ and $g$ are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$. Suppose $Z_f$ and $Z_g$ are the number of zeros inside $\gamma$, and $P_f$ and $P_g$ are the number of poles inside $\gamma$ (counted according to their multiplicities) and that $|f(z) + g(z)| < |f(z)| + |g(z)|$ on $\gamma$. Then $Z_f - P_f = Z_g - P_g$.

**Proof.** By hypothesis, $\left|\frac{f(z)}{g(z)} + 1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$ on $\gamma$ ($g$ has no zeros on $\gamma$, by hypothesis). If $\lambda = f(z)/g(z)$ for some given $z \in \{\gamma\}$ and $\lambda$ is a nonnegative real number, then the inequality becomes $\lambda + 1 < \lambda + 1$, a contradiction.

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# Theorem V.3.8 (continued)

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**Proof (continued).** So

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{f/g} \text{ by Corollary IV.1.22} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{g}{f}\frac{f'g - fg'}{g^2} = \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right) \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} \\ &= (Z_f - P_f) - (Z_g - P_g) \text{ by the Argument Principle.} \quad \square \end{aligned}$$