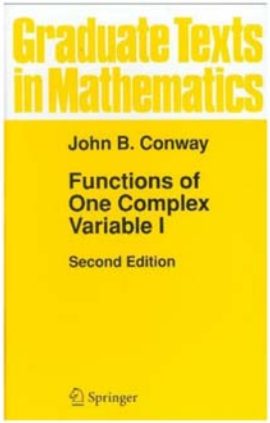
Complex Analysis

Chapter VI. The Maximum Modulus Theorem

VI.1. The Maximum Principle—Proofs of Theorems



Theorem VI.1.2

Theorem VI.1.2. Maximum Modulus Theorem—Second Version. Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\overline{G}$ which is analytic in G. Then

$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}.$$

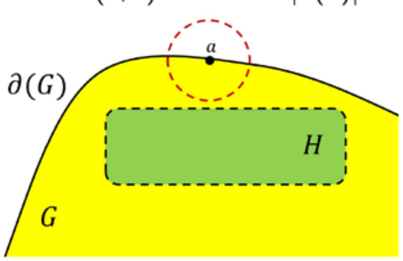
($\overline{G}$ is G closure and ∂G is the boundary of G.)

Proof. Since $\overline{G}$ is bounded and closed, there is $a \in \overline{G}$ such that $|f(a)| \geq |f(z)|$ (a continuous function on a compact set). If f is constant, the result is trivial; if f is not constant the result follows from the Maximum Modulus (Theorem IV.3.11) applied to each open component of G, say $\mathcal{O}_i$ for $i \in \mathbb{N}$, and then observing that $\partial(G) = \partial(\cup\mathcal{O}_i)$. □

Theorem VI.1.4

Theorem VI.1.4. Maximum Modulus Theorem—Third Version. Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a} |f(z)| \leq M$ for all a in $\partial_\infty(G)$. Then $|f(z)| \leq M$ for all z in G.

Proof. Let $\delta > 0$ and define $H = \{z \in G \mid |f(z)| > M + \delta\}$. We want to show $H = \varnothing$. Since $|f|$ is continuous (and real valued) then H is open (H is the inverse image of an open set). Since $\limsup_{z\to a} |f(z)| \leq M$ for each $a \in \partial_\infty(G)$, there is some $B(a;r)$ such that $|f(z)| < M + \delta$ for all $z \in G \cap B(a;r)$:



Theorem VI.1.4 (continued)

Proof (continued). So any closure point of H is not "near" the boundary of G and $\overline{H} \subset G$. Since this also holds for unbounded G with $a = \infty$, H must be bounded (if G is bounded then H is bounded since $H \subset G$; if G is unbounded then $\infty \in \partial_\infty(G)$ and there exists $R > 0$ such that $H \cap (\mathbb{C} \setminus B(a;R)) \cap G) = \varnothing$). Therefore $\overline{H}$ is compact. So the Maximum Modulus Theorem—Second Version (Theorem VI.1.2) applies to f on H and $\max\{|f(z)| \mid z \in \overline{H}\} = \max\{|f(z)| \mid z \in \partial(H)\}$. Since $\overline{H} \subset \{z \mid |f(z)| \geq M + \delta\}$, for $z \in \partial(H)$ we have $|f(z)| = M + \delta$ by the continuity of f, since $|f(z)| \leq M + \delta$ for $z \in G \setminus \overline{H}$. So by the Maximum Modulus Theorem—Second Version (Theorem VI.1.2), $H = \varnothing$ or f is constant. But by the definition of H, if f is constant, then $H = \varnothing$. So we conclude that H must be empty and the result follows. □

Theorem VI.1.D

Theorem VI.1.D. Maximum Modulus Theorem for Unbounded Domains 2.
Let $z(t)$, $t \in [\alpha, \beta]$, define a Jordan curves Γ with its trace in $\mathbb{C}$ (that is, Γ is a simple closed curve in $\mathbb{C}$), and denote the open interior of Γ by Ω. Also, let φ be a function which is analytic in $\mathbb{C} \setminus \{\Gamma \cup \Omega\}$ and continuous on $\mathbb{C} \setminus \Omega$ such that $|\varphi(z)| \le M$ for all $z \in \Gamma$. Suppose, in addition, that $\varphi(z)$ tends to a finite limit ℓ as z tends to infinity and set $\varphi(\infty) = \ell$. Then, $|\varphi(z)| \le M$ for all x in $\mathbb{C}_\infty \setminus \{\Gamma \cap \Omega\}$, unless φ is a constant.

Proof. Let $a \in \Omega$ and denote by Γ_a the Jordan curve $z(t) - a$, $t \in [\alpha, \beta]$. Then the open interior of Γ_a contains the origin 0. Let γ_a be the Jordan curve $w(t) = 1/(z(t) - a)$, $t \in [\alpha, \beta]$, and let Δ_a be the open interior of γ_a. Notice that for $x \in \gamma_a$,

$$\varphi(a + w^{-1}) = \varphi\left(a + \left(\frac{1}{z(t) - a}\right)^{-1}\right) = \varphi(z(t)) \in \varphi(\Gamma). \qquad (*)$$

Theorem VI.1.D (continued)

Proof (continued). Because of the inversion in $w(t) = 1/(z(t) - a)$, for w in the open exterior of Γ, $\varphi(w)$ is in the open interior of γ_a (and $w = 0$ corresponds to the point $\infty \in \mathbb{C}_\infty$ through the inversion). So the function $\Phi(w) = \varphi(a + w^{-1})$ is analytic in the open interior of γ_a, except at 0 (and notice that from $(*)$, $\Phi(\gamma_a) = \varphi(\Gamma)$). That is, $\Phi(w)$ is analytic on $\Delta_a \setminus \{0\}$. Since, by hypothesis, $\lim_{z\to\infty} \varphi(z) = \ell$ then $\lim_{w \to 0} \Phi(w) = \ell$. So, defining $\Phi(0) = \ell$, Φ becomes analytic throughout Δ_a. Also, Φ is continuous on $\gamma_a \cup \Delta_a$ and $|\Phi(w)| \le M$ for w on γ_a (since $\Phi(w) = \varphi(z)$ for some z on Γ) by hypothesis. So, by the Maximum Modulus Theorem—Second Version (Theorem VI.1.2), $|\Phi(w)| \le M$ for all $w \in \gamma_a \cup \Delta_a$. By the Maximum Modulus Theorem (Theorem IV.3.11), either $|\Phi(w)| < M$ for all $w \in \Delta_a$ (the open interior of γ_a) or Φ is a constant function. Since $\Phi(w) = \varphi(a + w^{-1})$, and $\Phi(\gamma_a) = \varphi(\Gamma)$ then $|\varphi(z)| < M$ for all $z \in \mathbb{C}_\infty \setminus \{\Gamma \cup \Omega\}$ (that is, for z exterior to γ, including ∞ where $\varphi(\infty) = \Phi(0) = \ell$). □