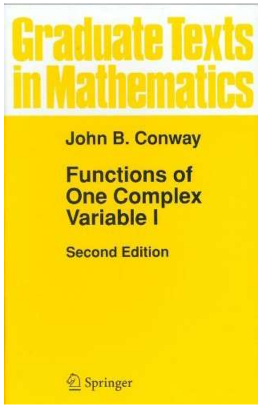
Complex Analysis

Chapter VI. The Maximum Modulus Theorem

Supplement. Applications of the Maximum Modulus Theorem to Polynomials—Proofs of Theorems



The Centroid Theorem

The Centroid Theorem.
The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.

Proof. Let polynomial p have zeros $z_1, z_2, \ldots, z_n$. Then $p(z) = \sum_{k=0}^n a_k z^k = a_n \prod_{k=1}^n (z - z_k)$. Multiplying out, we find that the coefficient of z^{n-1} is $a_{n-1} = -a_n(z_1 + z_2 + \cdots + z_n)$. Therefore the centroid of the zeros of p is

$$\frac{z_1 + z_2 + \cdots + z_n}{n} = \left(\frac{1}{n}\right)\left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Let the zeros of p' be $w_1, w_2, \ldots, w_{n-1}$. Then

$$p'(z) = \sum_{k=1}^n k a_k z^{k-1} = na_n \prod_{k=1}^{n-1}(z - w_k).$$

The Centroid Theorem (continued)

The Centroid Theorem.
The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.

Proof (continued). Multiplying out, we find that the coefficient of z^{n-2} is

$$(n-1)a_{n-1} = -na_n(w_1 + w_2 + \cdots + w_{n-1}).$$

Therefore the centroid of the zeros of p' is

$$\frac{w_1 + w_2 + \cdots + w_{n-1}}{n-1} = \left(\frac{1}{n-1}\right)\left(\frac{-(n-1)a_{n-1}}{na_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Therefore the centroid of the zeros of p' is the same as the centroid of the zeros of p. □

The Lucas Theorem

The Lucas Theorem.
If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.

Proof. By the Fundamental Theorem of Algebra, we can factor p as $p(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$. So

$$\log p(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)$$

and differentiating both sides gives

$$\frac{p'(z)}{p(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^n \frac{1}{z - r_k}. \quad (1)$$

The Lucas Theorem (continued 1)

Proof (continued). Suppose the half-plane H that contains all the zeros of $p(z)$ is described by $\text{Im}((z-a)/b) \le 0$. Then

$$\text{Im}((r_1-a)/b) \le 0,\ \text{Im}((r_2-a)/b) \le 0,\ \ldots,\ \text{Im}((r_n-a)/b) \le 0.$$

Now let z^* be some number not in H. We want to show that $p'(z^*) \ne 0$ (this will mean that all the zeros of $p'(z)$ are in H). Well, $\text{Im}((z^*-a)/b) > 0$. Let r_k be some zero of p. Then

$$\text{Im}\left(\frac{z^*-r_k}{b}\right) = \text{Im}\left(\frac{z^*-a-r_k+a}{b}\right) = \text{Im}\left(\frac{z^*-a}{b}\right) - \text{Im}\left(\frac{r_k-a}{b}\right) > 0.$$

(Notice that $\text{Im}((z^*-a)/b) > 0$ since z^* is not in H and $-\text{Im}((r_k-a)/b) \ge 0$ since r_k is in H.) The imaginary parts of reciprocal numbers have opposite signs, so $\text{Im}(b/(z^*-r_k)) < 0$.

The Lucas Theorem (continued 2)

The Lucas Theorem.
If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.

Proof (continued). Recall

$$\frac{p'(z)}{p(z)} = \frac{1}{z-r_1} + \frac{1}{z-r_2} + \cdots + \frac{1}{z-r_n} = \sum_{k=1}^{n} \frac{1}{z-r_k}. \quad (1)$$

Applying (1),

$$\text{Im}\left(\frac{bp'(z^*)}{p(z^*)}\right) = \sum_{k=1}^{n} \text{Im}\left(\frac{b}{z^*-r_k}\right) < 0.$$

So $p'(z^*)/p(z^*) \ne 0$ and $p'(z^*) \ne 0$. Therefore if $p'(z) = 0$ then $z \in H$. □

The Eneström-Kakeya Theorem

The Eneström-Kakeya Theorem.
If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with coefficients satisfying

$$0 \le a_0 \le a_1 \le \cdots \le a_n,$$

then all the zeros of p lie in $|z| \le 1$.

Proof. Define f by the equation $p(z)(1-z) = a_0 + (a_1-a_0)z + (a_2-a_1)z^2 + \cdots + (a_n-a_{n-1})z^n - a_n z^{n+1} = f(z) - a_n z^{n+1}$. Then for $|z| = 1$, we have

$$\begin{aligned} |f(z)| &\le |a_0| + |a_1-a_0| + |a_2-a_1| + \cdots + |a_n-a_{n-1}| \\ &= a_0 + (a_1-a_0) + (a_2-a_1) + \cdots + (a_n-a_{n-1}) = a_n. \end{aligned}$$

Notice that the function $z^n f(1/z) = \sum_{j=0}^{n}(a_j - a_{j-1})z^{n-j}$ (where we take $a_{-1} = 0$) has the same bound on $|z| = 1$ as f. Namely, $|z^n f(1/z)| \le a_n$ for $|z| = 1$. Since $z^n f(1/z)$ is analytic in $|z| \le 1$, we have $|z^n f(1/z)| \le a_n$ for $|z| \le 1$ by the Maximum Modulus Theorem. Hence, $|f(1/z)| \le a_n/|z|^n$ for $|z| \le 1$.

The Eneström-Kakeya Theorem (continued)

The Eneström-Kakeya Theorem.
If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with coefficients satisfying

$$0 \le a_0 \le a_1 \le \cdots \le a_n,$$

then all the zeros of p lie in $|z| \le 1$.

Proof (continued). Hence, $|f(1/z)| \le a_n/|z|^n$ for $|z| \le 1$. Replacing z with $1/z$, we see that $|f(z)| \le a_n z^n$ for $|z| \ge 1$, and making use of this we get, $|(1-z)p(z)| = |f(z) - a_n z^{n+1}| \ge a_n|z|^{n+1} - |f(z)| \ge a_n|z|^{n+1} - a_n|z|^n = a_n|z|^n(|z|-1)$. So if $|z| > 1$ then $(1-z)p(z) \ne 0$. Therefore, all the zeros of p lie in $|z| \le 1$. □

Rate of Growth Theorem

Rate of Growth Theorem (Bernstein).
If p is a polynomial of degree n such that $|p(z)| \le M$ on $|z| = 1$, then for $R \ge 1$ we have

$$\max_{|z|=R} |p(z)| \le MR^n.$$

Proof. For $p(z) = \sum_{k=0}^n a_k z^k$ we have $r(z) = z^n p(1/z) = \sum_{k=0}^n a_k z^{n-k}$. Notice that for $|z| = 1$ (and $1/z = \overline{z}$) we have $\|r\| = \|p\|$ where $\|p\| = \max_{|z|=1} |p(z)|$. By the Maximum Modulus Theorem, for $|z| \le 1$ we have $|r(z)| \le \|r\| = \|p\| \le M$. That is, $|z^n p(1/z)| \le M$ for $|z| \le 1$. Replacing z with $1/z$, we have $|(1/z^n)p(z)| \le M$ for $|z| \ge 1$, or $|p(z)| \le M|z|^n$ for $|z| \ge 1$. □

Bernstein Lemma

Bernstein Lemma. Let p and q be polynomials such that (i) $\lim_{|z|\to\infty} |p(z)/q(z)| \le 1$, (ii) $|p(z)| \le |q(z)|$ for $|z| = 1$, and (iii) all zeros of q lie in $|z| \le 1$. Then $|p'(z)| \le |q'(z)|$ for $|z| = 1$.

Proof. Define $f(z) = p(z)/q(z)$. Then f is analytic on $|z| > 1$, $|f(z)| \le 1$ for $|z| = 1$, and $\lim_{|z|\to\infty} |f(z)| \le 1$. So by the Maximum Modulus Principle for Unbounded Domains,

$$|f(z)| \le 1 \text{ for } |z| \ge 1. \qquad (*)$$

Let $|\lambda| > 1$ and define polynomial $g(z) = p(z) - \lambda q(z)$. If $g(z_0) = p(z_0) - \lambda q(z_0) = 0$ and if $q(z_0) \ne 0$ then $|p(z_0)| = |\lambda||q(z_0)| > |q(z_0)|$. Therefore $|f(z_0)| = |p(z_0)/q(z_0)| > 1$ and so $|z_0| < 1$ by $(*)$. Now if $q(z_0) = 0$, then $|z_0| \le 1$ and it could be that $|z_0| = 1$ in which case $p(z_0) = 0$ and $g(z_0) = 0$. So all zeros of g lie in $|z| \le 1$.

Bernstein Lemma (continued)

Bernstein Lemma. Let p and q be polynomials such that (i) $\lim_{|z|\to\infty} |p(z)/q(z)| \le 1$, (ii) $|p(z)| \le |q(z)|$ for $|z| = 1$, and (iii) all zeros of q lie in $|z| \le 1$. Then $|p'(z)| \le |q'(z)|$ for $|z| = 1$.

Proof (continued). By Lucas' Theorem, g' has all its zeros in $|z| \le 1$. So for no $|\lambda| > 1$ is $g'(z) = p'(z) - \lambda q'(z) = 0$ where $|z| > 1$; or in other words, $p'(z)/q'(z) = \lambda$ where $|\lambda| > 1$ has no solution in $|z| > 1$. Hence $|p'(z)| \le |q'(z)|$ for $|z| > 1$. By taking limits, we have $|p'(z)| \le |q'(z)|$ for $|z| \ge 1$, and the result follows. □

Bernstein's Inequality

Bernstein's Inequality.
Let p be a polynomial of degree n. Then

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|.$$

Proof. Let $M = \max_{|z|=1} |p(z)|$ and define $q(z) = Mz^n$. Then (i) $|p(z)| \le R^n M$ for $|z| = R$ by Bernstein's Rate of Growth Theorem, and so $\lim_{|z|=R\to\infty} |p(z)/q(z)| \le \lim_{R\to\infty} (R^n M)/(R^n M) = 1$, (ii) $|p(z)| \le |q(z)| = M$ on $|z| = 1$, and (iii) all zeros of q lie in $|z| \le 1$. So, by the Bernstein's Lemma, $|p'(z)| \le |q'(z)|$ for $|z| = 1$. This implies that

$$\max_{|z|=1} |p'(z)| \le \max_{|z|=1} |q'(z)| = \max_{|z|=1} |nMz^{n-1}| = nM = n \max_{|z|=1} |p(z)|.$$

□