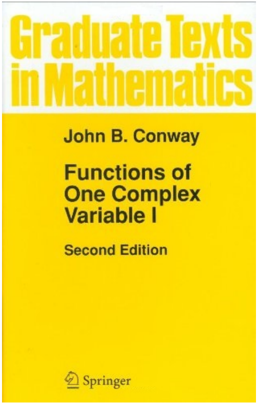
Complex Analysis

Chapter VI. The Maximum Modulus Theorem

Supplement. Applications of the Maximum Modulus Theorem to Polynomials—Proofs of Theorems



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The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.

Proof. Let polynomial p have zeros $z_1, z_2, \ldots, z_n$. Then $p(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k)$. Multiplying out, we find that the coefficient of z^{n-1} is $a_{n-1} = -a_n(z_1 + z_2 + \cdots + z_n)$.

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$$\frac{z_1 + z_2 + \cdots + z_n}{n} = \left(\frac{1}{n}\right)\left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Let the zeros of p' be $w_1, w_2, \ldots, w_{n-1}$. Then

$$p'(z) = \sum_{k=1}^{n} ka_k z^{k-1} = na_n \prod_{k=1}^{n-1}(z - w_k).$$

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Therefore the centroid of the zeros of p' is

$$\frac{w_1 + w_2 + \cdots + w_{n-1}}{n-1} = \left(\frac{1}{n-1}\right)\left(\frac{-(n-1)a_{n-1}}{na_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Therefore the centroid of the zeros of p' is the same as the centroid of the zeros of p. □

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The Lucas Theorem

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If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.

Proof. By the Fundamental Theorem of Algebra, we can factor p as $p(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$. So

$$\log p(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)$$

and differentiating both sides gives

$$\frac{p'(z)}{p(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^{n} \frac{1}{z - r_k}. \quad (1)$$

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Proof (continued). Suppose the half-plane H that contains all the zeros of $p(z)$ is described by $\text{Im}((z-a)/b) \leq 0$. Then

$$\text{Im}((r_1-a)/b) \leq 0, \text{Im}((r_2-a)/b) \leq 0, \ldots, \text{Im}((r_n-a)/b) \leq 0.$$

Now let z^* be some number not in H. We want to show that $p'(z^*) \neq 0$ (this will mean that all the zeros of $p'(z)$ are in H). Well, $\text{Im}((z^*-a)/b) > 0$. Let r_k be some zero of p.

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$$\text{Im}\left(\frac{z^* - r_k}{b}\right) = \text{Im}\left(\frac{z^* - a - r_k + a}{b}\right) = \text{Im}\left(\frac{z^* - a}{b}\right) - \text{Im}\left(\frac{r_k - a}{b}\right) > 0.$$

(Notice that $\text{Im}((z^* - a)/b) > 0$ since z^* is not in H and $-\text{Im}((r_k - a)/b) \geq 0$ since r_k is in H.) The imaginary parts of reciprocal numbers have opposite signs, so $\text{Im}(b/(z^* - r_k)) < 0$.

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If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.

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Applying (1),

$$\mathrm{Im}\left(\frac{bp'(z^*)}{p(z^*)}\right) = \sum_{k=1}^{n} \mathrm{Im}\left(\frac{b}{z^* - r_k}\right) < 0.$$

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If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with coefficients satisfying

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n,$$

then all the zeros of p lie in $|z| \leq 1$.

Proof. Define f by the equation $p(z)(1-z) = a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} = f(z) - a_n z^{n+1}$. Then for $|z| = 1$, we have

$$\begin{aligned} |f(z)| &\leq |a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| \\ &= a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n. \end{aligned}$$

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Notice that the function $z^n f(1/z) = \sum_{j=0}^{n} (a_j - a_{j-1}) z^{n-j}$ (where we take $a_{-1} = 0$) has the same bound on $|z| = 1$ as f. Namely, $|z^n f(1/z)| \leq a_n$ for $|z| = 1$. Since $z^n f(1/z)$ is analytic in $|z| \leq 1$, we have $|z^n f(1/z)| \leq a_n$ for $|z| \leq 1$ by the Maximum Modulus Theorem. Hence, $|f(1/z)| \leq a_n / |z|^n$ for $|z| \leq 1$.

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Rate of Growth Theorem

Rate of Growth Theorem (Bernstein).
If p is a polynomial of degree n such that $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have

$$\max_{|z|=R} |p(z)| \leq MR^n.$$

Proof. For $p(z) = \sum_{k=0}^{n} a_k z^k$ we have $r(z) = z^n p(1/z) = \sum_{k=0}^{n} a_k z^{n-k}$. Notice that for $|z| = 1$ (and $1/z = \overline{z}$) we have $\|r\| = \|p\|$ where $\|p\| = \max_{|z|=1} |p(z)|$. By the Maximum Modulus Theorem, for $|z| \leq 1$ we have $|r(z)| \leq \|r\| = \|p\| \leq M$. That is, $|z^n p(1/z)| \leq M$ for $|z| \leq 1$. Replacing z with $1/z$, we have $|(1/z^n)p(z)| \leq M$ for $|z| \geq 1$, or $|p(z)| \leq M|z|^n$ for $|z| \geq 1$. □

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Let $|\lambda| > 1$ and define polynomial $g(z) = p(z) - \lambda q(z)$. If $g(z_0) = p(z_0) - \lambda q(z_0) = 0$ and if $q(z_0) \ne 0$ then $|p(z_0)| = |\lambda||q(z_0)| > |q(z_0)|$. Therefore $|f(z_0)| = |p(z_0)/q(z_0)| > 1$ and so $|z_0| < 1$ by $(*)$. Now if $q(z_0) = 0$, then $|z_0| \le 1$ and it could be that $|z_0| = 1$ in which case $p(z_0) = 0$ and $g(z_0) = 0$. So all zeros of g lie in $|z| \le 1$.

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Bernstein's Inequality

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Let p be a polynomial of degree n. Then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

Proof. Let $M = \max_{|z|=1} |p(z)|$ and define $q(z) = Mz^n$. Then
(i) $|p(z)| \leq R^n M$ for $|z| = R$ by Bernstein's Rate of Growth Theorem, and so $\lim_{|z|=R\to\infty} |p(z)/q(z)| \leq \lim_{R\to\infty}(R^n M)/(R^n M) = 1$,
(ii) $|p(z)| \leq |q(z)| = M$ on $|z| = 1$, and (iii) all zeros of q lie in $|z| \leq 1$. So, by the Bernstein's Lemma, $|p'(z)| \leq |q'(z)|$ for $|z| = 1$. This implies that

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