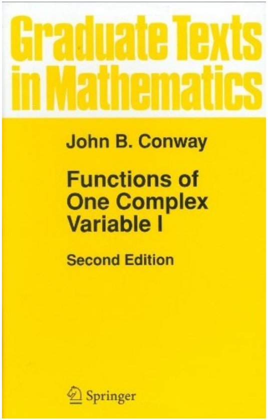
Complex Analysis

Chapter VI. The Maximum Modulus Theorem

VI.1. The Maximum Principle—Proofs of Theorems



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$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}.$$

($\overline{G}$ is G closure and ∂G is the boundary of G.)

Proof. Since $\overline{G}$ is bounded and closed, there is $a \in \overline{G}$ such that $|f(a)| \geq |f(z)|$ (a continuous function on a compact set). If f is constant, the result is trivial; if f is not constant the result follows from the Maximum Modulus (Theorem IV.3.11) applied to each open component of G, say $\mathcal{O}_i$ for $i \in \mathbb{N}$, and then observing that $\partial(G) = \partial(\cup \mathcal{O}_i)$. □

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Theorem VI.1.4

Theorem VI.1.4. Maximum Modulus Theorem—Third Version.
Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z \to a} |f(z)| \le M$ for all a in $\partial_\infty(G)$. Then $|f(z)| \le M$ for all z in G.

Proof. Let $\delta > 0$ and define $H = \{z \in G \mid |f(z)| > M + \delta\}$. We want to show $H = \varnothing$. Since $|f|$ is continuous (and real valued) then H is open (H is the inverse image of an open set).

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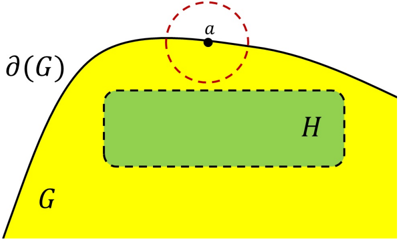

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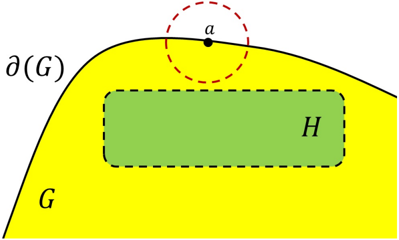

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Proof (continued). So any closure point of H is not "near" the boundary of G and $\overline{H} \subset G$. Since this also holds for unbounded G with $a = \infty$, H must be bounded (if G is bounded then H is bounded since $H \subset G$; if G is unbounded then $\infty \in \partial_\infty(G)$ and there exists $R > 0$ such that $H \cap (\mathbb{C} \setminus B(a; R)) \cap G) = \varnothing$). Therefore $\overline{H}$ is compact. So the Maximum Modulus Theorem—Second Version (Theorem VI.1.2) applies to f on H and $\max\{|f(z)| \mid z \in \overline{H}\} = \max\{|f(z)| \mid z \in \partial(H)\}$. Since $\overline{H} \subset \{z \mid |f(z)| \geq M + \delta\}$, for $z \in \partial(H)$ we have $|f(z)| = M + \delta$ by the continuity of f, since $|f(z)| \leq M + \delta$ for $z \in G \setminus \overline{H}$. So by the Maximum Modulus Theorem—Second Version (Theorem VI.1.2), $H = \varnothing$ or f is constant.

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Let $z(t)$, $t \in [\alpha, \beta]$, define a Jordan curves Γ with its trace in $\mathbb{C}$ (that is, Γ is a simple closed curve in $\mathbb{C}$), and denote the open interior of Γ by Ω. Also, let φ be a function which is analytic in $\mathbb{C} \setminus \{\Gamma \cup \Omega\}$ and continuous on $\mathbb{C} \setminus \Omega$ such that $|\varphi(z)| \leq M$ for all $z \in \Gamma$. Suppose, in addition, that $\varphi(z)$ tends to a finite limit ℓ as z tends to infinity and set $\varphi(\infty) = \ell$. Then, $|\varphi(z)| \leq M$ for all x in $\mathbb{C}_\infty \setminus \{\Gamma \cap \Omega\}$, unless φ is a constant.

Proof. Let $a \in \Omega$ and denote by Γ_a the Jordan curve $z(t) - a$, $t \in [\alpha, \beta]$. Then the open interior of Γ_a contains the origin 0. Let γ_a be the Jordan curve $w(t) = 1/(z(t) - a)$, $t \in [\alpha, \beta]$, and let Δ_a be the open interior of γ_a.

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