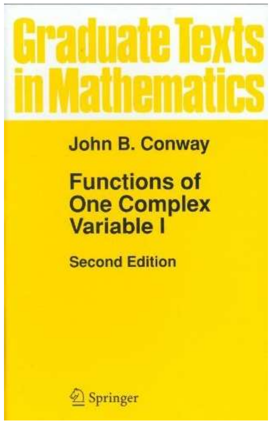
Complex Analysis

Chapter VI. The Maximum Modulus Theorem

VI.2. Schwarz's Lemma—Proofs of Theorems



Lemma VI.2.1

Lemma VI.2.1. Schwarz's Lemma.
Let $D = \{z \mid |z| < 1\}$ and suppose f is analytic on D with
(a) $|f(z)| \le 1$ for $z \in D$, and
(b) $f(0) = 0$.
Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all z in the disk D. Moreover if $|f'(z)| = 1$ of if $|f(z)| = |z|$ for some $z \ne 0$ then there is a constant $c \in \mathbb{C}$, $|c| = 1$, such that $f(w) = cw$ for all $w \in D$.

Proof. Define $g : D \to \mathbb{C}$ as $g(z) = \begin{cases} f(z)/z & \text{for } z \ne 0 \\ f'(0) & \text{for } z = 0. \end{cases}$ Since $f(0) = 0$, then $f(z) = zh(z)$ for some $h(z)$ analytic on D by Corollary IV.3.9. Notice that $g(z) = f(z)/z = h(z)$ for $z \ne 0$. Also,

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) = g(0),$$

so g is continuous at $z = 0$ and, since $h(z)$ is analytic on D it is also continuous on D, so $g(z) = h(z)$ for all $z \in D$.

Lemma VI.2.1 (continued)

Proof (continued). Therefore, g is analytic on D. For any $0 < r < 1$, we have by hypothesis that for $|z| \le r$, $|g(z)| = |f(z)/z| \le 1/r$, and so by the Maximum Modulus Theorem (Second Version—Theorem VI.1.1), $|g(z)| \le 1/r$ for $|z| \le r$ and $0 < r < 1$. Letting r approach 1 gives $|g(z)| \le 1$ for all $z \in D$. Therefore, $|f(z)| \le |z|$ for $z \in D$ and $|f'(0)| = |g(0)| \le 1$. If $|f(z)| = |z|$ for some $z \in D$, $z \ne 0$, or $|f'(0)| = |g(0)| = 1$ then $|g|$ assumes its maximum value inside D. Then, by the Maximum Modulus Theorem (Theorem VI.1.1), $g(z) = c$ for some constant $c \in \mathbb{C}$ with $|c| = 1$. Then, since $g(z) = f(z)/z$, we have $f(z) = cz$ for some $|c| = 1$ and for all $z \in D$. □

Proposition VI.2.2

Proposition VI.2.2. If $|a| < 1$ then φ_a is a one to one map of the open unit disk D onto itself. The inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Proof. The one to one and onto claim is established above by the existence of an inverse of φ_a. The fact that $\varphi_a^{-1} = \varphi_{-a}$ is also established above.

For $z \in \partial D$ we have $z = e^{i\theta}$, and

$$|\varphi_a(z)| = |\varphi_a(e^{i\theta}| = \left| \frac{e^{i\theta} - a}{1 - \overline{a}e^{i\theta}} \right| = \frac{|e^{i\theta} - a|}{|1 - \overline{a}e^{i\theta}|} \frac{1}{|e^{-i\theta}|}$$

$$= \frac{|e^{i\theta} - a|}{|e^{-i\theta} - \overline{a}|} = \frac{|e^{i\theta} - a|}{|e^{i\theta} - a|} = 1.$$

So $\varphi_a(\partial D) \in \partial D$. Since φ_a is a Möbius transformation, it is one to one and onto ∂D.

Proposition VI.2.2 (continued)

Proposition VI.2.2. If $|a| < 1$ then φ_a is a one to one map of the open unit disk D onto itself. The inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Proof (continued). Finally, $\varphi_a(a)\dfrac{(a) - (a)}{1 - \overline{a}(a)} = 0$ and

$$\varphi_2'(0) = \frac{[1](1 - \overline{a}(0)) - ((0) - a)[-\overline{a}]}{(1 - \overline{a}(0))^2} = 1 - |a|^2.$$

Also,

$$\varphi_a'(a) = \frac{[1](1 - \overline{a}(a)) - ((a) - a)[-\overline{a}]}{(a - \overline{a}(a))^2} = \frac{1 - |a|^2}{(1 - |a|)^2} = (1 - |a|^2)^{-1}.$$

□

Lemma VI.2.A

Lemma VI.2.A. Suppose f is analytic on $D = \{z \mid |z| < 1\}$ and $|f(z)| \le 1$ for $z \in D$. Let $z \in D$. Then

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

Moreover, equality holds exactly when $f(z) = \varphi_\alpha(c\varphi_a(z))$, where $\alpha = f(a)$ for some $c \in \mathbb{C}$ where $|c| = 1$.

Proof. Let $g(z) = \varphi_\alpha \circ f \circ \varphi_{-a}(z)$ where $\alpha = f(a)$. Then g maps D into D and $g(0) = \varphi_\alpha(f(\varphi_{-a}(0))) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0$. So by the Schwarz's Lemma applied to g, $|g'(0)| \le 1$. From the Chain Rule,

$$\begin{aligned} g'(0) &= (\varphi_\alpha \circ f)'(\varphi_{-a}(0))[\varphi_{-a}'(0)] \\ &= (\varphi_\alpha \circ f)'(a)(1 - |a|^2) \text{ since } \varphi_{-a}(z) = \frac{z + a}{1 - \overline{a}z}, \\ & \qquad \varphi_{-a}'(z) = \frac{[1](1 + \overline{a}z) - (z - a)[\overline{a}]}{(1 + \overline{a}z)^2} = \frac{1 - |z|^2}{(1 + \overline{a}z)^2} \end{aligned}$$

Lemma VI.2.A (continued 1)

Proof (continued).

$$\begin{aligned} & \text{and } \varphi_{-a}'(0) = 1 - |a|^2 \\ &= \varphi_\alpha'(f(a))[f'(a)](1 - |a|^2) = \varphi_\alpha'(\alpha)f'(a)(1 - |a|^2) \\ &= \frac{1 - |a|^2}{1 - |\alpha|^2} f'(a) \text{ since } \varphi_\alpha'(z) = \frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)^2} \\ & \text{and } \varphi_\alpha'(\alpha) = \frac{1 - |\alpha|^2}{(1 - |\alpha|^2)^2} = \frac{1}{1 - |\alpha|^2} \\ &= \frac{1 - |a|^2}{1 - |f(a)|^2} f'(a). \end{aligned}$$

Since $|g'(0)| \le 1$, we have

$$|g'(0)| = \left|\frac{1 - |a|^2}{1 - |f(a)|^2} f'(a)\right| \le 1, \text{ or } |f'(a)| \le \left|\frac{1 - |f(a)|^2}{1 - |a|^2}\right| = \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

Lemma VI.2.A (continued 2)

Lemma VI.2.A. Suppose f is analytic on $D = \{z \mid |z| < 1\}$ and $|f(z)| \le 1$ for $z \in D$. Let $z \in D$. Then

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

Moreover, equality holds exactly when $f(z) = \varphi_\alpha(c\varphi_a(z))$, where $\alpha = f(a)$ for some $c \in \mathbb{C}$ where $|c| = 1$.

Proof (continued). "Moreover," we have equality by Schwarz's Lemma exactly when $g(z) = cz$ for some $c \in \mathbb{R}$, $|c| = 1$. That is,

$$g(z) = cz = \varphi_\alpha \circ f \circ \varphi_{-a}(z), \text{ or } \varphi_\alpha^{-1}(cz) = f \circ \varphi_{-a}(z),$$

$$\text{or } \varphi_{-\alpha}(cz) = f \circ \varphi_{-a}(z) \text{ since } \varphi_\alpha^{-1} = \varphi_{-\alpha},$$

or, replacing z with $\varphi_a(z)$, $\varphi_{-\alpha}(c\varphi_z(z)) = f \circ \varphi_{-z}(\varphi_a(z)) = f(z)$. □

Theorem VI.2.5

Theorem VI.2.5. Let $f : D \to D$ be a one to one analytic map of D onto itself and suppose $f(a) = 0$. Then there is a complex c where $|c| = 1$ such that $f = c\varphi_a$.

Proof. Since f is one to one and onto, then there is $g : D \to D$ such that $g(f(z)) = z$ for $|z| < 1$ and g is analytic by Proposition III.2.20. Applying Lemma VI.2.A to both f and g gives

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2} = (1 - |a|^2)^{-1} \text{ since } f(a) = 0, \qquad (*)$$

and $|g'(0)| \le \dfrac{1 - |g(0)|^2}{1 - |0|^2} = 1 - |a|^2$ since $f(a) = 0$, and so $g(0) = a$. But $z = g(f(z))$ and $1 = g'(f(z))f'(z)$ and, in particular for $z = a$, $1 = g'(f(a))f'(a) = g'(0)f'(a)$. So $|f'(a)| = 1/|g'(0)| \ge (1 - |a|^2)^{-1}$, which combines with $(*)$ to give $|f'(a)| = (1 - |a|^2)^{-1}$.

Theorem VI.2.5 (continued)

Theorem VI.2.5. Let $f : D \to D$ be a one to one analytic map of D onto itself and suppose $f(a) = 0$. Then there is a complex c where $|c| = 1$ such that $f = c\varphi_a$.

Proof (continued). So by the second conclusion in Lemma VI.2.A, we have that

$$\begin{aligned} f(z) &= \varphi_{-\alpha}(c\varphi_a(z)) = \varphi_0(c\varphi_a(z)) \text{ since } \alpha = f(a) = 0 \\ &= c\varphi_a(z) \text{ since } \varphi_0(z) = z \end{aligned}$$

where $c \in \mathbb{R}$, $|c| = 1$. □

Generalized Schwarz's Lemma 1

Generalized Schwarz's Lemma 1.
If f is analytic on $\overline{D}\{z \mid |z| \le 1\}$, with
(a) $|f(z)| \le M$ for $z \in \overline{D}$, and
(b) $f(a) = 0$ where $|a| < 1$.
Then for $z \in \overline{D}$:

$$|f(z)| \le M \left| \frac{z - a}{a - \overline{a}z} \right| = M|\varphi_a(z)|.$$

Proof. Define $g(z) = f \circ \varphi_a^{-1}(z) = f \circ \varphi_{-a}(z) = f((z + a)/(1 + \overline{a}z))$. We know that for $z \in D$ we have $|g(z)| = |f((z - a)/(1 - \overline{a}z))| \le M$ and $g(0) = f(a) = 0$. So the function $g(z)/M$ satisfies the hypotheses of Schwarz's Lemma and we have that $|g(z)|/M \le |z|$ for $z \in \overline{D}$, or $|g(z)| \le M|z|$ or $|f((z - a)/(1 - \overline{a}z))| \le M|z|$. Replacing z with $\varphi_a(z)$ to get

$$|f(z)| \le M|\varphi_a(z)| = M \left| \frac{z - a}{1 - \overline{a}z} \right|. \quad \square$$