Complex Analysis

Chapter VI. The Maximum Modulus Theorem

VI.3. Convex Functions and Hadamard's Three Circles Theorem—Proofs of Theorems

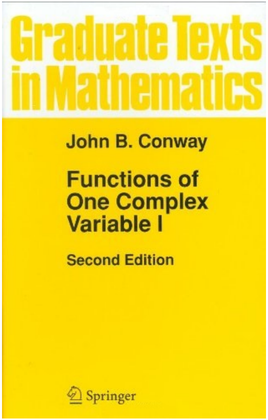

Table of contents

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Proposition VI.3.2. A function $f : [a, b] \to \mathbb{R}$ is convex if and only if the set

$$A = \{(x, y) \mid x \in [a, b] \text{ and } f(x) \le y\}$$

is convex.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is convex. Let $(x_1, y_1), (x_2, y_2), \in A \subset \mathbb{R}^2$. If $t \in [0, 1]$ then (by the definition of convex function f)

$$f(tx_2 + (1 - t)x_1) \le tf(x_2) + (1 - t)f(x_1) \le ty_2 + (1 - t)y_1$$

where $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

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where $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus

$$t(x_2, y_2) + (1-t)(x_1, y_1) = (t_2x_2 + (1-t)x_1, ty_2 + (1-t)y_1) \in A$$

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Proof (continued). Suppose A is a convex set and let $x_1, x_2 \in [1, b]$. Then

$$(tx_2 + (1 - t)x_1, tf(x_2) + (1 - t)f(x_2)) \in A$$

for all $t \in [0, 1]$. But the definition of set A implies that

$$f(tx_2 + (1 - t)x_1) \leq tf(x_2) + (1 - t)f(x_1).$$

So f satisfies the definition of convexity, and hence f is convex. □

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Proposition VI.3.4

Proposition VI.3.4. A differentiable function f on $[a, b]$ is convex if and only if f' is increasing.

Proof. Suppose f is convex. Let $x, y \in [a, b]$ with $x < y$ and suppose $t \in [0, 1]$.

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Proof. Suppose f is convex. Let $x, y \in [a, b]$ with $x < y$ and suppose $t \in [0, 1]$. Since $0 < t(x - y) = (1 - t)x + ty - x$, the convexity of f implies that $f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$, and

$$\frac{f((1-t)x+ty)}{t(y-x)} \leq \frac{(1-t)f(x)+tf(y)}{t(y-x)} = \frac{f(x)}{t(y-x)} + \frac{t(f(y)-f(x))}{t(y-x)},$$

or

$$\frac{f((1-t)x+ty)-f(x)}{t(y-x)} \leq \frac{f(y)-f(x)}{y-x},$$

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$$\frac{f(x+t(y-x))-f(x)}{t(y-x)} \leq \frac{f(y)-f(x)}{y-x}.$$

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Now as $t \to 0$, $x + t(y - x) \to x$ and $t(y - x) \to 0$ (and these last two limits occur at the same rate), so we have $f'(x) \leq \dfrac{f(y)-f(x)}{y-x}$.

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Proposition VI.3.4 (continued 1)

Proof (continued). Also, $0 > (1-t)(x-y) = (1-t)x + ty - y$, so $f((1-t)x+ty) \leq (1-t)f(x) + tf(y)$ and

$$\frac{f((1-t)x+ty)}{(1-t)(x-y)} \geq \frac{(1-t)f(x)+tf(y)}{(1-t)(x-y)}$$

$$= \frac{(1-t)f(x)}{(1-t)(x-y)} + \frac{tf(y)-f(y)+f(y)}{(1-t)(x-y)},$$

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$$\frac{f((1-t)x+ty)-f(y)}{(1-t)(x-y)} \geq \frac{f(x)-f(y)}{x-y},$$

or

$$\frac{f((1-t)(x-y)+y)-f(y)}{(1-t)(x-y)} \geq \frac{f(x)-f(y)}{x-y}.$$

Now as $t \to 1$, $(1-t)(x-y)+y \to y$, and $(1-t)(x-y) \to 0$ (and these last two limits occur at the same rate), so we have $f'(y) \geq \dfrac{f(x)-f(y)}{x-y}$.

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Now as $t \to 1$, $(1-t)(x-y)+y \to y$, and $(1-t)(x-y) \to 0$ (and these last two limits occur at the same rate), so we have $f'(y) \geq \dfrac{f(x)-f(y)}{x-y}$.

Proposition VI.3.4 (continued 2)

Proof (continued). Combining the above information, $f'(x) \leq \dfrac{f(x) - f(y)}{x - y} \leq f(y)$, and f' is increasing on $[a, b]$.

Now suppose that f' is increasing and that $x < u < y$. By the Mean Value Theorem, there are r, s where $x < r < u < s < y$ where $f'(r) = \dfrac{f(u) - f(x)}{u - x}$ and $f'(s) = \dfrac{f(y) - f(u)}{y - u}$. Since f' is increasing, then $f'(r) \leq f'(s)$ and so $\dfrac{f(u) - f(x)}{u - x} \leq \dfrac{f(y) - f(u)}{y - u}$ and this holds for any u where $x < u < y$.

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$f'(x) \leq \dfrac{f(x) - f(y)}{x - y} \leq f(y)$, and f' is increasing on $[a, b]$.

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$(1 - t)(f(u) - f(x)) \leq t(f(y) - f(u))$. So
$(1 - t)\{f((1 - t)x + ty) - f(x)\} \leq t\{f(y) - f((1 - t)x + ty)\}$ or
$f((1 - t)x + ty) - f(x) + tf(x) \leq tf(y)$ or
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Lemma VI.3.10

Lemma VI.3.10. Let f and G be as Theorem 3.7 and further suppose that $|f(z)| \leq 1$ for $z \in \partial G$. Then $|f(z)| \leq 1$ for $z \in G$.

Proof. Let $\varepsilon > 0$. Define $g_\varepsilon(z) = (1 + \varepsilon(z - a))^{-1}$ for $a \in G^-$. The for $z = x + iy \in G^-$,

$$|g_\varepsilon(z)| = \frac{1}{|a + \varepsilon(z - a)|} \leq \frac{1}{|\text{Re}(1 + \varepsilon(z - a))|} = \frac{1}{|\text{Re}(a + \varepsilon(x + iy - a))|} =$$

So for $z \in \partial G$ we have that $|f(z)g_\varepsilon(z)| \leq (1)(1) = 1$.

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So for $z \in \partial G$ we have that $|f(z)g_\varepsilon(z)| \leq (1)(1) = 1$. Since f is bounded by B in G (by the Theorem VI.3.7 hypothesis), we have

$$|f(z)g_\varepsilon(z)| \leq B|1 + \varepsilon(z - a)|^{-1} \leq B|\mathrm{Im}(1 + \varepsilon(z - a)|^{-1}$$
$$= B|\mathrm{Im}(\varepsilon z)|^{-1} = B|\varepsilon \mathrm{Im}(z)|^{-1}. \qquad (3.11)$$

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So if $R = \{x + iy \mid z \leq x \leq b, |y| < B/\varepsilon\}$, then inequality (3.11) gives for $z \in \partial R$:

$$|f(z)g_\varepsilon(z)| \leq B|\varepsilon \mathrm{Im}(z)|^{-1} = \frac{B}{\varepsilon|\mathrm{Im}(z)|} = \frac{B}{\varepsilon(B/\varepsilon)} = 1.$$

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Lemma VI.3.10. Let f and G be as Theorem 3.7 and further suppose that $|f(z)| \leq 1$ for $z \in \partial G$. Then $|f(z)| \leq 1$ for $z \in G$.

Proof (continued). Then by the Maximum Modulus Theorem—Second Version (Theorem VI.1.2), $|f(z)g_\varepsilon(z)| \leq 1$ for all $z \in R$.

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Proof (continued). Then by the Maximum Modulus Theorem—Second Version (Theorem VI.1.2), $|f(z)g_\varepsilon(z)| \leq 1$ for all $z \in R$. Next, for $z \in G$ with $|\text{Im}(z)| > B/\varepsilon$, inequality (3.11) implies that

$$|f(z)g_\varepsilon(z)| \leq \frac{B}{\varepsilon|\text{Im}(z)|} < \frac{B}{\varepsilon(B/\varepsilon)} = 1.$$

So for all $z \in G$, $|f(z)g_\varepsilon(z)| \leq 1$ and $|f(z)| \leq 1/|g_\varepsilon(z)| \leq |1 + \varepsilon(z - a)|$. Since $\varepsilon > 0$ is arbitrary, we have $|f(z)| \leq 1$ for all $z \in G$. □

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Theorem VI.3.7

Theorem VI.3.7. Let $a < b$ and let G be the vertical strip $\{x + iy \mid a < x < b\}$. Suppose $f : G^- \to \mathbb{C}$ is continuous and f is analytic in G. If we define $M : [a, b] \to \mathbb{R}$ by

$$M(x) = \sup\{|f(x + iy)| \mid -\infty < y < \infty\}$$

and $|f(z)| < B$ for all $z \in G$, then $\log M(x)$ is a convex function.

Proof. By Exercise VI.3.3(c), f is convex if and only if
$\dfrac{f(u) - f(x)}{u - x} \leq \dfrac{f(y) - f(u)}{y - u}$, or
$(y - u)(f(u) - f(x)) \leq (u - x)(f(y) - f(u))$, or
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$(y - x)f(u) \le (y - u)f(x) + (u - x)f(y)$. With $f(z) = \log M(x)$, we have

$$(y - x)\log M(u) \le (y - u)\log M(x) + (u - x)\log M(y). \qquad (*)$$

Exponentiating both sides gives $M(u)^{y-x} \le M(x)^{y-u}M(Y)^{u-x}$ where $z \le x < u < y \le b$.

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Exponentiating both sides gives $M(u)^{y-x} \le M(x)^{y-u}M(Y)^{u-x}$ where $z \le x < u < y \le b$.

Theorem VI.3.7 (continued 1)

Proof (continued). So to prove Theorem VII.3.7 we must show that $M(u)^{b-a} \leq M(z)^{b-u} M(y)^{u-a}$ for all $u \in (a, b)$. Define $g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)}$. Then g is entire and nonzero (since $A^z = \exp(z \log A)$, so g is basically an exponential function). Since $|A^z| = A^{\text{Re}(z)}$, then for $z = x + iy$ we have

$$|g(z)| = M(a)^{(b-x)/(b-a)} M(b)^{(x-a)/(b-a)}. \qquad (3.12)$$

(Here, we assume $M(a) \neq 0$ and $M(b) \neq 0$ without loss of generality, since if either is 0 then either $f \equiv 0$ on the line $\text{Re}(z) = a$ or $f \equiv 0$ on the line $\text{Re}(z) = b$, and in both cases, by the Maximum Modulus Theorem—Third Version [Theorem VI.1.4], $f \equiv 0$ on G.)

Theorem VI.3.7 (continued 1)

Proof (continued). So to prove Theorem VII.3.7 we must show that $M(u)^{b-a} \leq M(z)^{b-u} M(y)^{u-a}$ for all $u \in (a, b)$. Define $g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)}$. Then g is entire and nonzero (since $A^z = \exp(z \log A)$, so g is basically an exponential function). Since $|A^z| = A^{\mathrm{Re}(z)}$, then for $z = x + iy$ we have

$$|g(z)| = M(a)^{(b-x)/(b-a)} M(b)^{(x-a)/(b-a)}. \qquad (3.12)$$

(Here, we assume $M(a) \neq 0$ and $M(b) \neq 0$ without loss of generality, since if either is 0 then either $f \equiv 0$ on the line $\mathrm{Re}(z) = a$ or $f \equiv 0$ on the line $\mathrm{Re}(z) = b$, and in both cases, by the Maximum Modulus Theorem—Third Version [Theorem VI.1.4], $f \equiv 0$ on G.) The right hand side of (3.12) is a continuous function of x for $x \in [a, b]$, and since it is an exponential function it is nonzero. So it attains a minimum (and a maximum) and $|g|^{-1}$ is therefore bounded on G^-.

Theorem VI.3.7 (continued 1)

Proof (continued). So to prove Theorem VII.3.7 we must show that $M(u)^{b-a} \leq M(z)^{b-u}M(y)^{u-a}$ for all $u \in (a,b)$. Define $g(z) = M(a)^{(b-z)/(b-a)}M(b)^{(z-a)/(b-a)}$. Then g is entire and nonzero (since $A^z = \exp(z \log A)$, so g is basically an exponential function). Since $|A^z| = A^{\text{Re}(z)}$, then for $z = x + iy$ we have

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Theorem VI.3.7 (continued 2)

Proof (continued). Also, $|g(a+iy)| = M(a)$ an d$|g(b+iy)| = M(b)$, so for $\text{Re}(z) = \text{Re}(x_i y) = x = a$, we have

$$\left|\frac{f(z)}{g(z)}\right| \le \frac{M(a)}{M(a)^{(b-a)/(b-a)}M(b)^{(a-a)/(b-a)}} = 1$$

and for $\text{Re}(z) = \text{Re}(x+iy) = x = b$, we have

$$\left|\frac{f(z)}{g(z)}\right| \le \frac{M(b)}{M(a)^{(b-b)/(b-a)}M(a)^{(b-a)/(b-a)}} = 1.$$

So for $z \in \partial G^-$, $|f(z)/g(z)| \le 1$. Now Lemma VI.3.10 holds and implies that $|f(z)/g(z)| \le 1$ for all $z \in G$, or that $|f(z)| \le |g(z)|$ for all $z \in G$. With $z \in G$, $z = u_i v$ (so $a < u < b$), and from (3.12) we have $|f(z)| \le M(u) \le M(a)^{(b-u)/(b-a)}M(b)^{(u-a)/(b-a)} = |g(z)|$, or $M(u)^{b-a} \le M(a)^{b-u}M(b)^{u-a}$ for all $u \in (a,b)$. As stated above, this proves the claim. □

Theorem VI.3.7 (continued 2)

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Corollary VI.3.9

Corollary VI.3.9. Let $a < b$ and let G be the vertical strip $\{x + iy \mid a < x < b\}$. Let $f : G^- \to \mathbb{C}$ be continuous and let f be analytic on G. Then for all $z \in G$ we have

$$|f(z)| < \sup\{|f(z)| \mid z \in \partial G\}.$$

Proof. Since $\log M(x)$ is convex by Theorem VI.3.7, then for $z \in G$ we have

$$M(x) \leq \max\{M(a), M(b)\} = \sup\{|f(z)| \mid z \in \partial G^-\}.$$

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Theorem VI.3.13

Theorem VI.3.13. Hadamard's Three Circles Theorem.
Let $0 < R_1 < R_2 < \infty$ and suppose f is analytic and not identically zero on $\text{ann}(0; R_1, R_2)$. If $R_1 < r < R_2$, define

$$M(r) = \max\{|f(re^{i\theta})| \mid 0 \leq \theta \leq 2\pi\}.$$

Then for $R_1 < r_1 \leq r \leq r_2 < R_2$ and $r_1 \neq r_2$, we have

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2).$$

Proof. (This is Exercise VI.3.4.) First, define $g(z) = f(e^z)$. since f is analytic on $\text{ann}(0; R_1, R_2)$, then g is analytic on the vertical strip $\{x + iy \mid \log R_1 \leq x \leq \log R_2\}$. Since f is continuous on $\text{ann}(0; R_1, R_2)$, then f is bounded on the annulus (the annulus is compact). Therefore, g is bounded on the vertical strip. So g satisfies the hypothesis of Theorem VI.3.7.

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Theorem VI.3.13 (continued 1)

Proof (continued). Now for $z = x + iy$ in the vertical strip, we have e^z in the annulus and

$$M_g(z) = \sup\{|g(x+iy)| \mid -\infty < y < \infty\} = \sup\{|f(e^{x+iy})| \mid -\infty < y < \infty\}$$
$$= \max\{|f(e^x e^{i\theta})| \mid 0 \le \theta \le 2\pi\} = M_f(e^x).$$

Hence, for $\log R_1 < \log r_1 \le \log r \le \log r_2 < \log R_2$ where $r_1 \ne r_2$, we have by Theorem VI.3.7 applied to g (actually from equation $(*)$ in the proof of Theorem VI.3.7 [see page 136] with $x = \log r_1$, $a = \log r$, and $y = \log r_2$):

$$(\log r_2 - \log r_1) \log M_g(\log r) \le (\log r_2 - \log r) \log M_g(\log r_1)$$
$$+(\log r - \log r_1) \log M_g(\log r_2)$$

or

$$(\log r_2 - \log r_1) \log M_f(r) \le (\log r_2 - \log r) \log M_f(r_1)$$
$$= (\log r - \log r_1) \log M_f(r_2), \ldots$$

Theorem VI.3.13 (continued 1)

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$$= (\log r - \log r_1) \log M_f(r_2), \ldots$$

Theorem VI.3.13 (continued 2)

Theorem VI.3.13. Hadamard's Three Circles Theorem.
Let $0 < R_1 < R_2 < \infty$ and suppose f is analytic and not identically zero on $\text{ann}(0; R_1, R_2)$. If $R_1 < r < R_2$, define

$$M(r) = \max\{|f(re^{i\theta})| \mid 0 \leq \theta \leq 2\pi\}.$$

Then for $R_1 < r_1 \leq r \leq r_2 < R_2$ and $r_1 \neq r_2$, we have

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2).$$

Proof (continued). or

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2),$$

where $M(x)$ denotes $M_f(x)$. □