Complex Analysis

Chapter VI. The Maximum Modulus Theorem
VI.4. Phragmén-Lindelöf Theorem—Proofs of Theorems

Theorem VI.4.A

Theorem VI.4.A. Suppose f is an entire function, $M > 0$, and $0 < \alpha < 1$. Suppose $|f(z)| \le M + |z|^\alpha$ for all $z \in \mathbb{C}$. Then f is constant.

Proof. We take $n = 1$ in Corollary VI.2.13: $f'(a) = \dfrac{n!}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{(w-a)^2}\,dw$ where $\gamma(t) = a + re^{it}$ and $0 \le t \le 2\pi$. We have

$$\begin{aligned} |f'(a)| &= \left| \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^2}\,dw \right| \le \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|w-a|^2}\,|dw| \\ &= \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{r^2}\,|dw| \le \frac{n!}{2\pi r^2}\int_\gamma (M + |z|^\alpha)\,|dw| \\ &\le \frac{n!}{2\pi r^2}\int_\gamma (M + (|a|+r)^\alpha)\,|dw| \text{ since } |w| \le |a| + r \\ &= \frac{n!}{2\pi r^2} 2\pi r (M + (|a|+r)^\alpha). \end{aligned}$$

Theorem VI.4.A (continued)

Theorem VI.4.A. Suppose f is an entire function, $M > 0$, and $0 < \alpha < 1$. Suppose $|f(z)| \le M + |z|^\alpha$ for all $z \in \mathbb{C}$. Then f is constant.

Proof (continued). ...

$$|f'(a)| \le \frac{n!}{2\pi r^2} 2\pi r (M + (|a|+r)^\alpha).$$

Since this holds for arbitrary r (f is entire), then we see that $r \to \infty$ implies that $f'(a) = 0$. Also, a is arbitrary, so $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is constant. □

Proposition VI.4.1

Theorem VI.4.1. Phragmén-Lindelöf Theorem.
Let G be a simply connected region and let f be an analytic function on G. Suppose there is an analytic function $\phi : G \to \mathbb{C}$ which is nonzero and is bounded on G. If M is a constant and $\partial_\infty G = A \cup B$ such that

(a) for every $a \in A$ we have $\limsup_{z\to a} |f(z)| \le M$, and

(b) for every $b \in B$ and $\eta > 0$, we have $\limsup_{z\to b} |f(z)||\phi(z)|^\eta \le M$,

then $|f(z)| \le M$ for all $z \in G$.

Proof. Let $|\varphi(z)| \le \kappa$ for all $z \in G$. Since G is simply connected and φ is nonzero on G, then by Corollary IV.6.17, there is a branch of $\log \varphi(z)$ on G. Hence $g(z) = \exp(\eta \log \varphi(z))$ is an analytic branch of $(\varphi(z))^\eta$ for $\eta > 0$, and $|g(z)| = |\varphi(z)|^\eta$. Define $F : G \to \mathbb{C}$ as $F(z) = f(z)g(z)\kappa^{-\eta}$. Then F is analytic on G and $|F(z)| \le |f(z)|$. Now for $a \in \partial_\infty G$ for which condition (a) holds, we have $\limsup_{z\to a} |F(z)| \le \limsup_{z\to\infty} |f(z)| \le M$.

Proposition VI.4.1

Proof. For $b \in \partial_\infty G$ for which condition (b) holds, we have

$$\limsup_{z\to b} |F(z)| = \limsup_{z\to b} |f(z)g(z)\kappa^{-\eta}| = \kappa^{-\eta} \limsup_{z\to b} |f(z)||\varphi(z)|^\eta \le \kappa^{-\eta}M.$$

So F satisfies the hypotheses of the Maximum Modulus Theorem—Third Version (Theorem VI.1.4) with M of Theorem VI.1.4 replaced with $\max\{M, \kappa^{-\eta}M\}$ here. so, by Theorem VI.1.4, $|f(z)| \le \max\{M, \kappa^{-\eta}M\}$ for all $z \in G$. So

$$|f(z)| = \kappa^\eta \frac{|F(z)|}{|g(z)|} = \frac{\kappa^\eta}{|\varphi(z)|^\eta}|F(z)| \le \left(\frac{\kappa}{|\varphi(z)|}\right)^\eta \max\{M, \kappa^{-\eta}M\}$$

for all $z \in G$ and for all $\eta > 0$. Letting $\eta \to 0^+$ implies $|f(z)| \le M$ for all $z \in G$. □

Corollary VI.4.2

Corollary VI.4.2. Let $a \ge 1/2$ and let $G = \{z \mid |\arg(z)| < \pi/(2a)\}$. Suppose that f is analytic on G and suppose there is a constant M such that $\limsup_{z\to w} |f(z)| \le M$ for all $w \in \partial G$. If there are positive constants P and $b < a$ such that $|f(z)| \le P\exp(|z|^b)$ for all z with $|z|$ sufficiently large, <u>then</u> $|f(z)| \le M$ for all $z \in G$.

Proof. Let $0 < b < c < a$ and define $\varphi(z) = \exp(-z^c)$ for $z \in G$. If $z = re^{i\theta}$ where $|\theta| < \pi/(2a)$, then $\mathrm{Re}(z^c) = r^c\cos(c\theta)$. So for $z \in G$,

$$|\varphi(z)| = |\exp(-z^c)| = |\exp(\mathrm{Re}(-z^c))| = \exp(-r^c\cos(c\theta))$$

when $z = re^{i\theta}$. Since $c < a$, $\cos(c\theta) \ge \rho > 0$ for some ρ (since $c < a$ implies $c\theta < a\theta < a(\pi/(2a)) = \pi/2$ for $z \in G$). So $|\varphi(z)| = |\exp(-z^c)| = \exp(-r^c\cos(c\theta)) \le \exp(-r^c\rho)$ for all $z \in G$, and φ is bounded on G. Also, if $\eta > 0$ and $z = re^{i\theta}$ is sufficiently large, then

Corollary VI.4.2 (continued)

Proof (continued).

$$\begin{aligned} |f(z)||\varphi(z)|^\eta &\le P\exp(|z|^b)|\varphi(z)|^\eta \text{ by hypothesis} \\ &\le P\exp(|z|^b)(\exp(-r^c\rho))^\eta \\ &= P\exp(r^b - \eta r^c\rho) \text{ since } z = re^{i\theta} \end{aligned}$$

(the "sufficiently large" is required of $z = re^{i\theta}$ to get the bound $|f(z)| \le P\exp(|z|^b)$ for z "sufficiently large" as is hypothesized). But $r^b - \eta r^c\rho = r^c(r^{b-c} - \eta\rho)$. Since $b < c$, we have $b - c < 0$ and so $r^{b-c} \to 0^+$ as $r \to +\infty$. so we have $r^b - \eta r^c\rho - r^c(r^{b-c} - \eta\rho) \to -\infty$ as $r \to +\infty$. Now $\limsup_{z\to a} |f(z)| \le M$ for all $a \in \partial G$ by hypothesis. For $b = \infty$, $\limsup_z \to \infty ||f(z)||\varphi(z)|^\eta = \lim_{r\to\infty} P\exp(r^b - \eta r^c\rho) = 0 \le M$. So f satisfies the hypotheses of the Phragmén-Lindelöf Theorem (as does nonzero, analytic φ), and so $|f(z)| \le M$ for $z \in G$. □

Corollary VI.4.4

Corollary VI.4.4. Let $a \ge 1/2$ and let $G = \{z \mid \arg(z) < \pi/(2a)\}$, and suppose that for every $w \in \partial G$, $\limsup_{z\to w} |f(z)| \le M$. Moreover, assume that for every $\delta > 0$ there is a constant P (which may depend on δ) such that $|f(z)| \le P\exp(\delta|z|^a)$ for $z \in G$ and $|z|$ sufficiently large. <u>Then</u> $|f(z)| \le M$ for all $z \in G$.

Proof. Define $F : G \to \mathbb{C}$ as $F(z) = f(z)\exp(-\varepsilon z^1)$ where $\varepsilon > 0$ is fixed. If $x > 0$ and δ satisfies $0 < \delta < \varepsilon$ then there is a constant P with

$$\begin{aligned} |f(x)| &= |f(x)\exp(-\varepsilon x^a)| \\ &\le P\exp(\delta x^a)\exp(-\varepsilon x^a) \text{ for } x \text{ sufficiently large} \\ &= P\exp((\delta - \varepsilon)x^a). \end{aligned}$$

But then $|F(x)| \to 0$ as $x \to \infty$ ($x \in \mathbb{R}$). So $M_1 = \sup\{|F(x)| \mid 0 < x < \infty\} < \infty$ (since, say, $|F(x)| \le 1$ for x sufficiently large and then F is continuous on the complement of "sufficiently large" and so has a MAX there).

Corollary VI.4.4 (continued 1)

Proof (continued). Define $M_2 = \max\{M_1, M\}$ and
$H_+ = \{z \in G \mid 0 < \arg(z) < \pi/(2a)\}$,
$H_- = \{z \in G \mid -\pi/(2a) < \arg(z) < 0\}$. Notice that H_+ and H_- are sectors which share the boundary $\{x \mid 0 < x < \infty\}$. For any $w \in \partial H_- \cup \partial H_+$ with $|\arg(w)| = \pi/(2a)$ we have

$$\begin{aligned}
\limsup_{z\to w} |F(z)| &= \limsup_{z\to w} |f(z)||\exp(-\varepsilon z^a)| \\
&= \limsup_{z\to w} |f(z)|\exp(\text{Re }(-\varepsilon z^a)) \\
&= \limsup_{z\to w} |f(z)|\exp(-\varepsilon r^a \cos(a\theta)) \text{ for } z = re^{i\theta} \\
&\leq \limsup_{z\to w} |f(z)| \text{ since } -\varepsilon r^z \cos(a\theta) < 0 \\
&\qquad \text{because } |z\theta| < \pi/2 \\
&\leq M \text{ by hypothesis (i.e., definition of } M).
\end{aligned}$$

Corollary VI.4.4 (continued 2)

Proof (continued). So for all $w \in \partial H_- \cup \partial H_+$ we have $\limsup_{z\to w} |f(z)| \leq M_2$. In addition, we claim that the hypothesized condition $|f(z)| \leq P\exp(\delta|z|^a)$ for all $z \in G$ and $|z|$ sufficiently large implies that F satisfies the hypotheses of Corollary VI.4.2. (We have

$$|f(z)| = |f(z)\exp(-\varepsilon z^a)| \leq P\exp(\delta|z|^a)|\exp(-\varepsilon z^a)|$$

$$= P\exp(\delta|z|^a)\exp(\text{Re}(-\varepsilon z^a) = P\exp(\delta|z|^z - \varepsilon\text{Re}(z^a))$$

but this must be less than or equal to $P_1\exp(|z|^b)$ for positive P_1 and $0 < b < a$?) So...applying Corollary VI.4.2 to $F(z)$ gives $|F(z)| \leq M_2$ for all $z \in H_+ \cup H_-$. So $|f(z)| \leq M_2$ for all $z \in G$.

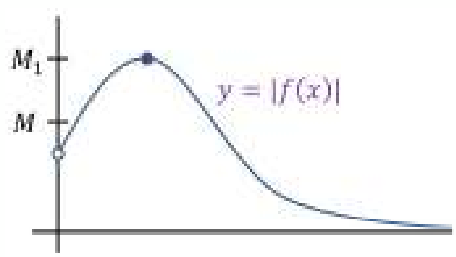
Corollary VI.4.4 (continued 3)

Proof (continued). We claim that $M_2 = M$. If not, then $M_1 = M_1 > M$.

But then we have that $|f(z)|$ assumes its maximum value in G at some positive real number $x \in G$ because $\lim_{x\to\infty} |F(x)| = 0$ as argued above and

$$\limsup_{x\to 0} |F(x)| = \limsup_{x\to 0} |f(x)||\exp(-\varepsilon x^a)|$$
$$= \limsup_{x\to 0} |f(x)|\exp(0) = \limsup_{x\to 0} |f(x)| \leq M < M_1,$$

so $|f|$ as a continuous function on $(0, \infty)$ must attain its supremum over this set:



Corollary VI.4.4 (continued 4)

Corollary VI.4.4. Let $a \geq 1/2$ and let $G = \{z \mid \arg(z) < \pi/(2a)\}$, and suppose that for every $w \in \partial G$, $\limsup_{z\to w} |f(z)| \leq M$. Moreover, assume that for every $\delta > 0$ there is a constant P (which may depend on δ) such that $|f(z)| \leq P\exp(\delta|z|^a)$ for $z \in G$ and $|z|$ sufficiently large. Then $|f(z)| \leq M$ for all $z \in G$.

Proof (continued). But then by the Maximum Modulus Theorem—First Version (Theorem VI.1.1), f must be a constant and then $M = M_1 = M_2$. So we have established that $M_2 = M$ and $|F(z)| \leq M$ for all $z \in G$. That is,

$$\begin{aligned}
|f(z)| &= |F(z)\exp(\varepsilon z^a)| \text{ by definition of } F \\
&\leq M\exp(\varepsilon\text{Re}(z^a))
\end{aligned}$$

for all $z \in G$. Since M is independent of ε, we can let arbitrary $\varepsilon \to 0$ and conclude that $|f(z)| \leq M$ for all $z \in G$. □