Complex Analysis

Chapter VI. The Maximum Modulus Theorem

VI.4. Phragmén-Lindelöf Theorem—Proofs of Theorems

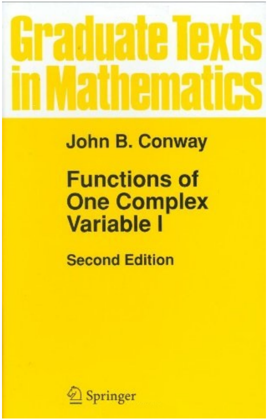

Table of contents

Theorem VI.4.A

Theorem VI.4.A. Suppose f is an entire function, $M > 0$, and $0 < \alpha < 1$. Suppose $|f(z)| \leq M + |z|^{\alpha}$ for all $z \in \mathbb{C}$. Then f is constant.

Proof. We take $n = 1$ in Corollary VI.2.13: $f'(a) = \dfrac{n!}{2\pi i} \displaystyle\int_{\gamma} \dfrac{f(w)}{(w-a)^2}\, dw$ where $\gamma(t) = a + re^{it}$ and $0 \leq t \leq 2\pi$.

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$$\begin{aligned}
|f'(a)| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^2}\,dw\right| \le \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|w-a|^2}\,|dw| \\
&= \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{r^2}\,|dw| \le \frac{n!}{2\pi r^2}\int_\gamma (M + |z|^\alpha)\,|dw| \\
&\le \frac{n!}{2\pi r^2}\int_\gamma (M + (|a|+r)^\alpha)\,|dw| \text{ since } |w| \le |a| + r \\
&= \frac{n!}{2\pi r^2} 2\pi r(M + (|a|+r)^\alpha).
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Proof (continued). ...

$$|f'(a)| \leq \frac{n!}{2\pi r^2} 2\pi r (M + (|a| + r)^\alpha).$$

Since this holds for arbitrary r (f is entire), then we see that $r \to \infty$ implies that $f'(a) = 0$. Also, a is arbitrary, so $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is constant. □

Proposition VI.4.1

Theorem VI.4.1. Phragmén-Lindelöf Theorem.
Let G be a simply connected region and let f be an analytic function on G. Suppose there is an analytic function $\phi : G \to \mathbb{C}$ which is nonzero and is bounded on G. If M is a constant and $\partial_\infty G = A \cup B$ such that

(a) for every $a \in A$ we have $\limsup_{z \to a} |f(z)| \leq M$, and

(b) for every $b \in B$ and $\eta > 0$, we have $\limsup_{z \to b} |f(z)||\phi(z)|^\eta \leq M$,

then $|f(z)| \leq M$ for all $z \in G$.

Proof. Let $|\varphi(z)| \leq \kappa$ for all $z \in G$. Since G is simply connected and φ is nonzero on G, then by Corollary IV.6.17, there is a branch of $\log \varphi(z)$ on G. Hence $g(z) = \exp(\eta \log \varphi(z))$ is an analytic branch of $(\varphi(z))^\eta$ for $\eta > 0$, and $|g(z)| = |\varphi(z)|^\eta$. Define $F : G \to \mathbb{C}$ as $F(z) = f(z)g(z)\kappa^{-\eta}$. Then F is analytic on G and $|F(z)| \leq |f(z)|$. Now for $a \in \partial_\infty G$ for which condition (a) holds, we have $\limsup_{z \to a} |F(z)| \leq \limsup_{z \to \infty} |f(z)| \leq M$.

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Proof. For $b \in \partial_\infty G$ for which condition (b) holds, we have

$$\limsup_{z \to b} |F(z)| = \limsup_{z \to b} |f(z)g(z)\kappa^{-\eta}| = \kappa^{-\eta} \limsup_{z \to b} |f(z)||\varphi(z)|^{\eta} \leq \kappa^{-\eta} M.$$

So F satisfies the hypotheses of the Maximum Modulus Theorem—Third Version (Theorem VI.1.4) with M of Theorem VI.1.4 replaced with $\max\{M, \kappa^{-\eta}M\}$ here. so, by Theorem VI.1.4, $|f(z)| \leq \max\{M, \kappa^{-\eta}M\}$ for all $z \in G$. So

$$|f(z)| = \kappa^{\eta} \frac{|F(z)|}{|g(z)|} = \frac{\kappa^{\eta}}{|\varphi(z)|^{\eta}} |F(z)| \leq \left(\frac{\kappa}{|\varphi(z)|} \right)^{\eta} \max\{M, \kappa^{-\eta}M\}$$

for all $z \in G$ and for all $\eta > 0$. Letting $\eta \to 0^{+}$ implies $|f(z)| \leq M$ for all $z \in G$. □

Corollary VI.4.2

Corollary VI.4.2. Let $a \geq 1/2$ and let $G = \{z \mid |\arg(z)| < \pi/(2a)\}$. Suppose that f is analytic on G and suppose there is a constant M such that $\limsup_{z\to w} |f(z)| \leq M$ for all $w \in \partial G$. If there are positive constants P and $b < a$ such that $|f(z)| \leq P\exp(|z|^b)$ for all z with $|z|$ sufficiently large, then $|f(z)| \leq M$ for all $z \in G$.

Proof. Let $0 < b < c < a$ and define $\varphi(z) = \exp(-z^c)$ for $z \in G$. If $z = re^{i\theta}$ where $|\theta| < \pi/(2a)$, then $\text{Re}(z^c) = r^c\cos(c\theta)$. So for $z \in G$,

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$$\begin{aligned} |f(z)||\varphi(z)|^\eta &\leq P\exp(|z|^b)|\varphi(z)|^\eta \text{ by hypothesis} \\ &\leq P\exp(|z|^b)(\exp(-r^c\rho))^\eta \\ &= P\exp(r^b - \eta r^c \rho) \text{ since } z = re^{i\theta} \end{aligned}$$

(the "sufficiently large" is required of $z = re^{i\theta}$ to get the bound $|f(z)| \leq P\exp(|z|^b)$ for z "sufficiently large" as is hypothesized). But $r^b - \eta r^c \rho = r^c(r^{b-c} - \eta\rho)$. Since $b < c$, we have $b - c < 0$ and so $r^{b-c} \to 0^+$ as $r \to +\infty$. so we have $r^b - \eta r^c \rho - r^c(r^{b-c} - \eta\rho) \to -\infty$ as $r \to +\infty$. Now $\limsup_{z\to a} |f(z)| \leq M$ for all $a \in \partial G$ by hypothesis. For $b = \infty$, $\limsup_z \to \infty ||f(z)||\varphi(z)|^\eta = \lim_{r\to\infty} P\exp(r^b - \eta r^c \rho) = 0 \leq M$.

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Proof (continued). Define $M_2 = \max\{M_1, M\}$ and
$H_+ = \{z \in G \mid 0 < \arg(z) < \pi/(2a)\}$,
$H_- = \{z \in G \mid -\pi/(2a) < \arg(z) < 0\}$. Notice that H_+ and H_- are sectors which share the boundary $\{x \mid 0 < x < \infty\}$. For any $w \in \partial H_- \cup \partial H_+$ with $|\arg(w)| = \pi/(2a)$ we have

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\limsup_{z \to w} |F(z)| &= \limsup_{z \to w} |f(z)||\exp(-\varepsilon z^a)| \\
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&= \limsup_{z \to w} |f(z)| \exp(-\varepsilon r^a \cos(a\theta)) \text{ for } z = re^{i\theta} \\
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Proof (continued). We claim that $M_2 = M$. If not, then $M_1 = M_1 > M$.

But then we have that $|f(z)|$ assumes its maximum value in G at some positive real number $x \in G$ because $\lim_{x\to\infty} |F(x)| = 0$ as argued above and

$$\limsup_{x\to 0} |F(x)| = \limsup_{x\to 0} |f(x)||\exp(-\varepsilon x^a)|$$

$$= \limsup_{x\to 0} |f(x)|\exp(0) = \limsup_{x\to 0} |f(x)| \le M < M_1,$$

so $|f|$ as a continuous function on $(0, \infty)$ must attain its supremum over this set:

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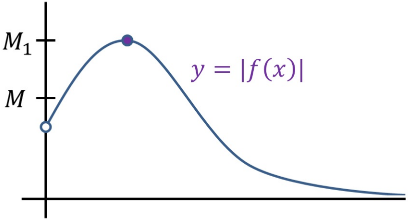

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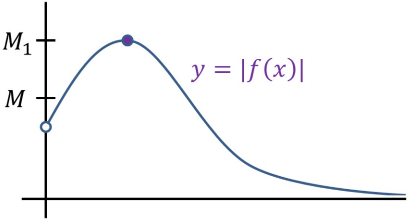
so $|f|$ as a continuous function on $(0, \infty)$ must attain its supremum over this set:



Corollary VI.4.4 (continued 4)

Corollary VI.4.4. Let $a \geq 1/2$ and let $G = \{z \mid \arg(z) < \pi/(2a)\}$, and suppose that for every $w \in \partial G$, $\limsup_{z \to w} |f(z)| \leq M$. Moreover, assume that for every $\delta > 0$ there is a constant P (which may depend on δ) such that $|f(z)| \leq P \exp(\delta |z|^a)$ for $z \in G$ and $|z|$ sufficiently large. Then $|f(z)| \leq M$ for all $z \in G$.

Proof (continued). But then by the Maximum Modulus Theorem—First Version (Theorem VI.1.1), f must be a constant and then $M = M_1 = M_2$. So we have established that $M_2 = M$ and $|F(z)| \leq M$ for all $z \in G$. That is,

$$\begin{aligned} |f(z)| &= |F(z) \exp(\varepsilon z^a)| \text{ by definition of } F \\ &\leq M \exp(\varepsilon \mathrm{Re}(z^a)) \end{aligned}$$

for all $z \in G$. Since M is independent of ε, we can let arbitrary $\varepsilon \to 0$ and conclude that $|f(z)| \leq M$ for all $z \in G$. □

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