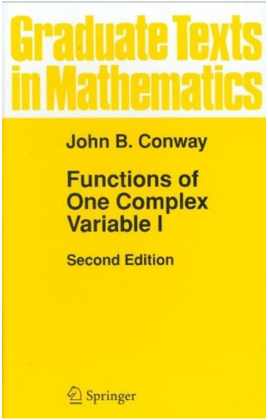
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.1. The Space of Continuous Functions $C(G, \Omega)$—Proofs of Theorems



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(a) $K_n \subset \text{int}(K_{n+1})$;

(b) $K \subset G$ and K compact imply $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty \setminus K_n$ contains a component of $\mathbb{C}_\infty \setminus G$.

Proof. For each $n \in \mathbb{N}$, let $K_n = \{z \mid |z| \leq n\} \cap \{z \mid d(z, \mathbb{C} \setminus G) \geq 1/n\}$. Since K_n is bounded (in modulus be n) and is closed (it's the intersection of two closed subsets of $\mathbb{C}$, then K_n is compact by the Heine-Borel Theorem (Theorem II.4.10).

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Theorem VII.1.2 (continued 1)

Proof (continued). As $n \to \infty$, $\{z \mid |z| \leq n\} \to \mathbb{C}$ and $\{z \mid d(z, \mathbb{C} \setminus G) \geq 1/n\} \to G$, so $G = \cup_{n=1}^{\infty} K_n$ and also $G = \cup_{n=1}^{\infty} \text{int}(K_n)$ (notice G is open). If K is a compact subset of G, then the sets $\text{int}(K_n)$ form an open cover of K and so has some finite subcover. Since the K_n are nested, then $K \subset K_n$ for some n and (b) follows.

For part (c), since K_n is bounded (in modulus by n), then $\mathbb{C} \setminus K_n$ has an unbounded component (notice that K_n may not be connected) which must contain ∞ (treated as a subset of $\mathbb{C}_\infty$) and since $K_n \subset G$ for all n then $\mathbb{C}_\infty \setminus K \supset \mathbb{C}_\infty \setminus G$ must be contained in the unbounded component of $\mathbb{C}_\infty \setminus K_n$.

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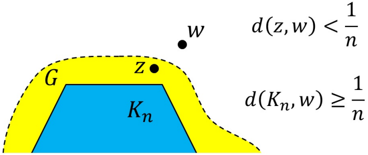
Proof (continued). So, by the definition of $d(z, \mathbb{C} \setminus G)$ in terms of an infimum, there must be $w \in \mathbb{C} \setminus G$ with $|w - z| < 1/n$. S0 $z \in B(w; 1/n)$. Now if $z' \in K_n$ then $d(z', \mathbb{C} \setminus G) \geq 1/n$ and since $w \in \mathbb{C} \setminus G$, we have $d(z', w) \geq 1/n$; hence $B(w, 1/n)$ contains no $z' \in K_n$. That is, $B(w, 1/n) \cap K_n = \varnothing$ and $B(w, 1/n) \subset \mathbb{C}_\infty \setminus K_n$.



Since disk $B(w, 1/n)$ is connected, $z \in B(w, 1/n)$, $z \in D$, and D is a connected component of $\mathbb{C}_\infty \setminus K_n$ then $B(w, 1/n) \subset D$. If D_1 is the component of $\mathbb{C}_\infty \setminus G$ that contains w, then $D_1 \subset D$ (since $G = \cup K_n$ then $\mathbb{C}_\infty \setminus G \subset \mathbb{C}_\infty \setminus K_n$ for each n). So component D of $\mathbb{C}_\infty \setminus K_n$ contains component D_1 of $\mathbb{C}_\infty \setminus G$ and (c) holds for bounded component of $\mathbb{C}_\infty \setminus K_n$. □

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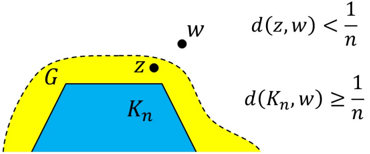

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Proposition VII.1.6

Proposition VII.1.6. $(C(G,\Omega),\rho)$ is a metric space.

Proof. Since d is a metric,

$$\rho_n(f,g) = \sup\{d(f(z),g(z)) \mid z \in K_n\}$$

$$= \sup\{d(g(z),f(z)) \mid z \in K_n\} = \rho_n(g,f)$$

and so

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^2 \frac{\rho_n(g,f)}{1+\rho_n(g,f)} = \rho(g,f).$$

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For the Triangle Inequality, let $f,g,h \in C(G,\Omega)$. Since d is a metric, for each $z \in K_n$ we have

$$d(f(z),g(z)) \le d(f(z),h(z)) + d(h(z),g(z)),$$

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Proof (continued). . . . so

$$\sup_{z\in K_n}\{d(f(z),g(z))\} \leq \sup_{z\in K_n}\{d(f(z),h(z))+d(h(z),g(z))\}$$

$$\leq \sup_{z\in K_n}\{d(f(z),h(z))\}+\sup_{z\in K_n}\{d(h(z),g(z)\}$$

or $\rho_n(f,g)\leq\rho_n(f,h)+\rho_n(h,g)$. So ρ_n is a metric on $C(G,\Omega)$. By Lemma VII.1.5, $\dfrac{\rho_n(f,g)}{1+\rho_n(f,g)}$ is also a metric on $C(G,\Omega)$ and so

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\leq\frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\frac{\rho_n(h,g)}{1+\rho_n(h,g)}.$$

Hence

$$\rho(f,g) \;=\; \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

Proposition VII.1.6 (continued 1)

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Hence

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Proposition VII.1.6 (continued 2)

Proof (continued). ... or

$$\begin{aligned}\rho(f,g) &\le \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\left(\frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\\ &\quad \text{by the absolute convergence}\\ &= \rho(f,h)+\rho(h,g).\end{aligned}$$

So the Triangle Inequality holds for ρ.

Finally, since $G = \cup_{n=1}^{\infty} K_n$ and $\rho_n(f,g) = 0$ if and only if $f(z) = g(z)$ for all $z \in K_n$, then $\rho(f,g) = 0$ if and only if $f(z) = g(z)$ for all $z \in G$ (i.e., $f \equiv g$ on G). So ρ is a metric (of course, ρ is nonnegative real valued). □

Proposition VII.1.6 (continued 2)

Proof (continued). ...or

$$\begin{aligned}
\rho(f,g) &\le \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\left(\frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\right)\\
&= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2\frac{\rho_n(h,g)}{1+\rho_n(h,g)}\\
&\qquad \text{by the absolute convergence}\\
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\end{aligned}$$

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Lemma VII.1.7

Lemma VII.1.7. Let the metric ρ be defined as above:

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

for $f,g \in C(G,\Omega)$ where $G = \cup_{n=1}^{\infty} K_n$ for compact K_n with $K_n \subset \text{int}(K_{n+1})$ and $\rho_n(f,g) = \sup\{d(f(z),g(z)) \mid z \in K_n\}$. If $\varepsilon > 0$ is given then there is $\delta > 0$ and a compact set $K \subset G$ such that for $f,g \in C(G,\Omega)$,

$$\sup\{d(f(z),g(z)) \mid z \in K\} < \delta \Longrightarrow \rho(f,g) < \varepsilon.$$

Conversely, if $\delta > 0$ and a compact set K are given then there is $\varepsilon > 0$ such that for $f,g \in C(G,\Omega)$,

$$\rho(f,g) < \varepsilon \Longrightarrow \sup\{d(f(z),g(z)) \mid z \in K\} < \delta.$$

Lemma VII.1.7 (continued 1)

Proof (continued). Let $\varepsilon > 0$ be fixed. Choose $p \in \mathbb{N}$ such that

$$\sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{1}{2}\varepsilon \qquad (*)$$

and put $K = K_p$. Choose $\delta > 0$ such that for all $0 \le t < \delta$ we have

$$\frac{t}{1+t} < \frac{1}{2}\varepsilon. \qquad (**)$$

Suppose $f, g \in C(G, \Omega)$ satisfy $\sup\{d(f(z), g(z)) \mid z \in K\} < \delta$. Since $K_n \subset K_p = K$ for $1 \le n \le p$ then $\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\} \le \delta$ for all $1 \le n \le p$. This gives $\dfrac{\rho_n(f,g)}{1+\rho_n(f,g)} \le \dfrac{1}{2}\varepsilon$ by $(**)$ for $1 \le n \le p$.

Lemma VII.1.7 (continued 1)

Proof (continued). Let $\varepsilon > 0$ be fixed. Choose $p \in \mathbb{N}$ such that

$$\sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{1}{2}\varepsilon \qquad (*)$$

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Lemma VII.1.7 (continued 2)

Proof (continued). Therefore

$$\begin{aligned} \rho(f,g) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \le \sum_{n=1}^{p} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\varepsilon\right) + \sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \text{ by } (*) \\ &= \varepsilon. \end{aligned}$$

Now suppose compact set K and $\delta > 0$ are given. Since $G = \cup_{n=1}^{\infty} K_n = \cup_{n=1}^{\infty} \text{int}(K_n)$ and K is compact, then there is $p \in \mathbb{N}$ such that $K \subset K_p$ (since $K \subset G$ is compact and $\{\text{int}(K_n) \mid n \in \mathbb{N}\}$ is an open cover of K). This gives

$$\rho_p(f,g) = \sup\{d(f(z),g(z)) \mid z \in K_p\} \ge \sup\{d(f(z),g(z)) \mid z \in K\}. \quad (***)$$

Let $\varepsilon > 0$ be such that $0 \le s < 2^p\varepsilon$ implies $s/(1-s) < \delta$.

Lemma VII.1.7 (continued 2)

Proof (continued). Therefore

$$\begin{aligned}\rho(f,g) &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \sum_{n=1}^{p}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\varepsilon\right) + \sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \text{ by } (*) \\ &= \varepsilon.\end{aligned}$$

Now suppose compact set K and $\delta > 0$ are given. Since $G = \cup_{n=1}^{\infty} K_n = \cup_{n=1}^{\infty}\text{int}(K_n)$ and K is compact, then there is $p \in \mathbb{N}$ such that $K \subset K_p$ (since $K \subset G$ is compact and $\{\text{int}(K_n) \mid n \in \mathbb{N}\}$ is an open cover of K). This gives

$$\rho_p(f,g) = \sup\{d(f(z),g(z)) \mid z \in K_p\} \geq \sup\{d(f(z),g(z)) \mid z \in K\}. \;(***)$$

Let $\varepsilon > 0$ be such that $0 \leq s < 2^p\varepsilon$ implies $s/(1-s) < \delta$.

Lemma VII.1.7 (continued 3)

Proof (continued). Then $t/(1+t) < 2^p\varepsilon$ implies (with $s = t/(1+t)$) that

$$\frac{t/(1+t)}{1-t/(1+t)} = \frac{t}{(1+t)-t} = t < \delta. \qquad (****)$$

So if $\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \varepsilon$, then with $n = p$ we have

$$\left(\frac{1}{2}\right)^p \frac{\rho_p(f,g)}{1+\rho_p(f,g)} < \varepsilon \text{ or } \frac{\rho_p(f,g)}{1+\rho_p(f,g)} < 2^p\varepsilon.$$

With $t = \rho_p(f,g)$ in $(****)$ we get that $\rho_p(f,g) < \delta$. Then by $(***)$ we have $\sup\{d(f(z),g(z)) \mid z \in K\} \leq \rho_p(f,g) < \delta$, as required. □

Lemma VII.1.7 (continued 3)

Proof (continued). Then $t/(1+t) < 2^p\varepsilon$ implies (with $s = t/(1+t)$) that

$$\frac{t/(1+t)}{1 - t/(1+t)} = \frac{t}{(1+t) - t} = t < \delta. \qquad (****)$$

So if $\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \varepsilon$, then with $n = p$ we have

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With $t = \rho_p(f,g)$ in $(****)$ we get that $\rho_p(f,g) < \delta$. Then by $(***)$ we have $\sup\{d(f(z),g(z)) \mid z \in K\} \le \rho_p(f,g) < \delta$, as required. □

Proposition VII.1.10

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(a) A set $\mathcal{O} \subset (C(G,\Omega),\rho)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that $\mathcal{O} \supset \{g \mid d(f(z),g(z)) < \delta \text{ for } z \in K\}$.

(b) A sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on all compact subsets of G.

Proof of (a). Set $\mathcal{O}$ in metric space $(C(G,\Omega),\rho)$ is open if for each $\{g \mid \rho(f,g) < \varepsilon\} \subset \mathcal{O}$. By Lemma VII.1.7, there is $\delta > 0$ and a compact K such that for $f,g \in C(G,\Omega)$ we have that $\sup\{d(f(z),g(z)) \mid z \in K\} < \delta$ implies $\rho(f,g) < \varepsilon$.

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(a) A set $\mathcal{O} \subset (C(G,\Omega),\rho)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that $\mathcal{O} \supset \{g \mid d(f(z),g(z)) < \delta \text{ for } z \in K\}$.

(b) A sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on all compact subsets of G.

Proof of (a). Set $\mathcal{O}$ in metric space $(C(G,\Omega),\rho)$ is open if for each $\{g \mid \rho(f,g) < \varepsilon\} \subset \mathcal{O}$. By Lemma VII.1.7, there is $\delta > 0$ and a compact K such that for $f,g \in C(G,\Omega)$ we have that $\sup\{d(f(z),g(z)) \mid z \in K\} < \delta$ implies $\rho(f,g) < \varepsilon$. Now for the given $f \in C(G,\Omega)$, if $g \in C(G,\Omega)$ satisfies $d(f(z),g(z)) < \delta$ for all $z \in K$ then $\sup\{d(f(z),g(z)) \mid z \in K\} < \delta$ (the inequality is still strict since K is compact) and then $\rho(f,g) < \varepsilon$. That is, $\{g \mid d(f(z),g(z)) < \delta \text{ for } z \in K\} \subset \mathcal{O}$.

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Proposition VII.1.10 (continued)

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(a) A set $\mathcal{O} \subset (C(G, \Omega), \rho)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that $\mathcal{O} \supset \{g \mid d(f(z), g(z)) < \delta \text{ for } z \in K\}$.

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Proof of (a), continued. Now suppose for each $f \in \mathcal{O}$ there is compact set K and $\delta > 0$ as stated. Then by the second part of Lemma VII.1.7, there is an $\varepsilon > 0$ such that for $g \in C(G, \Omega)$ we have that $\rho(f, g) < \varepsilon$ implies that $\sup\{d(f(z), g(z)) \mid z \in K\} < \delta$. So $g \in \{g \mid \rho(f, g) < \varepsilon\}$ implies $d(f(z), g(z)) < \delta$ for all $z \in K$ and so all such $g \in \mathcal{O}$ by hypothesis. That is, $\{g \mid \rho)g, f) < \varepsilon\} \subset \mathcal{O}$ and so $\mathcal{O}$ is open. □

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Proposition VII.1.12

Proposition VII.1.12. If metric space (Ω, d) is complete, then metric space $C(G, \Omega)$ is complete.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C(G, \Omega)$. For each component set $K \subset G$ the restrictions of the functions f_n to K gives a Cauchy sequence in $C(K, \Omega)$:

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) \mid z \in K_n\} \geq \rho_n(f|_K, g|_k) = \sup\{d(f(z), g(z)) \mid$$

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so $\rho(f, g) = \sum_{n=1}^{\infty} (1/2)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)} < \varepsilon$ implies

$$\rho(f|_K, g|_K) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f|_K, g|_K)}{1 + \rho_n(f|_K, g|_K)} < \rho(f, g) < \varepsilon$$

(since $\frac{d}{dt}\left[\frac{t}{1+t}\right] = \frac{1}{(1+t)^2} > 0$).

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Proposition VII.1.12 (continued 1)

Proof (continued). Now for any $\delta > 0$ and the given compact set K, by Lemma VII.1.7 part 2, there is $\varepsilon > 0$ such that for all $f, g \in C(G, \Omega)$ we have

$$\rho(f, g) < \varepsilon \text{ implies } \sup\{d(f(z), g(z)) \mid z \in K\} < \delta. \tag{1.13}$$

Since $\{f_n\}$ is a Cauchy sequence in $C(G, \Omega)$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\rho(f, g) < \varepsilon$ and so $\sup\{d(f_n(z), f_m(z)) \mid z \in K\} < \delta$. In particular, for each fixed $z \in K$, $\{f_n(z)\}$ is a Cauchy sequence in (Ω, d). Since (Ω, d) is complete, then $\{f_n(z)\}$ converges to some point in Ω; denote this point as $f(z)$. Then function f is defined on K.

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Proposition VII.1.12 (continued 2)

Proposition VII.1.12. If metric space (Ω, d) is complete, then metric space $C(G, \Omega)$ is complete.

Proof (continued). Let K be compact and fix $\delta > 0$. Choose $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\sup\{d(f_n(z), f_m(z)) \mid z \in K\} < \delta$ which can be done by (1.13). Let $z \in K$ be fixed. Since $\{f_n(z)\} \to f(z)$ as $n \to \infty$ (by choice of $f(z)$ in the first paragraph) then there is some $m \in \mathbb{N}$ with $m \geq N$ such that $d(f(z), f_m(z)) < \delta$. But then for all $n \geq N$ we have

$$d(f(z), f_n(z)) \leq d(f(z), f_m(z)) + d(f_m(z), f_n(z)) < \delta + \delta = 2\delta.$$

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Since N does not depend on z (but does depend on K), this gives that $\sup\{d(f(z), f_n(z)) \mid z \in K\} \to 0$ as $n \to \infty$. That is, $\{f_n\}$ converges to f uniformly on every compact set in G. So, in particular, $\{f_n\}$ converges to f uniformly on all closed balls contained in G. By Theorem II.6.1, f is continuous at each point of G. By Proposition VII.1.10(b), $\{f_n\}$ converges to f. □

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Proposition VII.1.16

Proposition VII.1.16. A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if for every compact set $K \subset G$ and every $\delta > 0$, there are functions $f_1, f_2, \ldots f_n \in \mathcal{F}$ such that for $f \in \mathcal{F}$ there is at least one k, $1 \leq k \leq n$, with

$$\sup\{d(f(z), f_k(z)) \mid z \in K\} < \delta.$$

Proof of "only if" part. Suppose $\mathcal{F}$ is normal and let compact $K \subset G$ and $\delta > 0$ be given. By Lemma VII.1.7 part 2, there exists $\varepsilon > 0$ such that for $f, g \in C(G, \Omega)$ we have

$$\rho(f, g) < \varepsilon \text{ implies } \sup\{d(f(z), g(z)) \mid z \in K\} < \delta. \qquad (*)$$

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But since $\mathcal{F}^-$ ($\mathcal{F}$ closure) is compact by Proposition VII.1.15, then $\mathcal{F}$ is totally bounded. (Actually, $\mathcal{F}^-$ is totally bounded by Theorem II.4.9; to show that $\mathcal{F}$ itself is totally bounded, GET THIS, Conway says "actually there are a few details to fill in here"! We move on and take this as given.)

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Proposition VII.1.16 (continued)

Proof (continued). So, by the definition of totally bounded (see page 22) there are $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subset \cup_{k=1}^{n} \{f \mid \rho(f, f_k) < \varepsilon\}$. From $(*)$, for each f_k we have
$\{f \mid \rho(f, f_k) < \varepsilon\} \subset \{f \mid d(f(z), f_k(z)) < \delta \text{ for } z \in K\}$.

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$$\mathcal{F} \subset \cup_{k=1}^{n} \{f \mid d(f(z), f_k(z)) < \delta \text{ for } z \in K\}.$$

So for any $f \in \mathcal{F}$, f must be in one of these sets on the right, say for $k = j$ and $d(f(z), f_j(z)) < \delta$ for all $z \in K$. Since K is compact,

$$\sup\{d(f(z), f_j(z)) < \delta, z \in K\} = \max\{d(f(z), f_j(z)) < \delta, z \in K\} < \delta.$$

So $\mathcal{F}$ satisfies the claimed condition. □

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Proposition VII.1.18

Proposition VII.1.18. The space $(\prod_{n=1}^{\infty} X_n, d)$ of the previous definition is a metric space. If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ if and only if $x_n^k \to x - n$ for all $n \in \mathbb{N}$. also, if each (X_n, d) is compact then X is compact.

Proof.

Claim 1. d is a metric. This proof is "left to the reader" (Exercise VII.1.3).

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Claim 1. d is a metric. This proof is "left to the reader" (Exercise VII.1.3).

Claim 2. If $\xi^k \to \xi$ then $x_n^k \to x_n$ for each $n \in \mathbb{N}$.

Suppose $d(\xi^k, \xi) \to 0$. Then

$$d(\xi^k, \xi) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \geq \left(\frac{1}{2}\right)^n \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \text{ for all } n \in \mathbb{N}$$

or $\dfrac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \leq 2^n d(\xi^k, \xi)$ for all $n \in \mathbb{N}$.

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Proposition VII.1.18 (continued 1)

Proof (continued). So $\lim_{k\to\infty} \frac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \le \lim_{k\to\infty} 2^n d(\xi^k, \xi) = 0$ and so $\lim_{k\to\infty} \frac{d_n(x_n^k, x_n)}{1 + d_x(x_n^k, x_n)} = 0$ for all $n \in \mathbb{N}$. Hence $\lim_{k\to\infty} d_n(x_n^k, x_n) = 0$ for all $n \in \mathbb{N}$, or $x_n^k \to x_n$ for all $n \in \mathbb{N}$.

Claim 3. If $x_n^k \to x_n$ for all $n \in \mathbb{N}$ then $\xi^k \to \xi$. The proof is "left to the reader" in Exercise VII.1.13.

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Claim 3. If $x_n^k \to x_n$ for all $n \in \mathbb{N}$ then $\xi^k \to \xi$. The proof is "left to the reader" in Exercise VII.1.13.

Claim 4. If each (X_n, d_n) is compact then X is compact. Suppose each (X_n, d_n) is compact. To show that (X, d) is compact it suffices to show that every sequence in X has a convergent subsequence (that is, it suffices to show that (X, d) is sequentially compact (see page 21) since this is equivalent to (X, d) being compact by Theorem II.4.9).

Proposition VII.1.18 (continued 1)

Proof (continued). So $\lim_{k\to\infty} \dfrac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \le \lim_{k\to\infty} 2^n d(\xi^k, \xi) = 0$ and so $\lim_{k\to\infty} \dfrac{d_n(x_n^k, x_n)}{1 + d_x(x_n^k, x_n)} = 0$ for all $n \in \mathbb{N}$. Hence $\lim_{k\to\infty} d_n(x_n^k, x_n) = 0$ for all $n \in \mathbb{N}$, or $x_n^k \to x_n$ for all $n \in \mathbb{N}$.

Claim 3. If $x_n^k \to x_n$ for all $n \in \mathbb{N}$ then $\xi^k \to \xi$. The proof is "left to the reader" in Exercise VII.1.13.

Claim 4. If each (X_n, d_n) is compact then X is compact. Suppose each (X_n, d_n) is compact. To show that (X, d) is compact it suffices to show that every sequence in X has a convergent subsequence (that is, it suffices to show that (X, d) is sequentially compact (see page 21) since this is equivalent to (X, d) being compact by Theorem II.4.9). We do so by the Cantor diagonalization process. Let $\xi^k = \{x_n^k\} \in X$ for $k \in \mathbb{N}$. Consider the sequence of the first entries of the ξ_k, that is $\{x_1^k\}_{k=1}^{\infty} \subset X_1$. Since X_1 is compact, and hence sequentially compact by Theorem II.4.9, there is a point $x_1 \in X_1$ and a subsequence of $\{x_1^k\}$ which converges to x_1.

Proposition VII.1.18 (continued 1)

Proof (continued). So $\lim_{k\to\infty} \dfrac{d_n(x_n^k, x_n)}{1 + d_n(x_n^k, x_n)} \le \lim_{k\to\infty} 2^n d(\xi^k, \xi) = 0$ and so $\lim_{k\to\infty} \dfrac{d_n(x_n^k, x_n)}{1 + d_x(x_n^k, x_n)} = 0$ for all $n \in \mathbb{N}$. Hence $\lim_{k\to\infty} d_n(x_n^k, x_n) = 0$ for all $n \in \mathbb{N}$, or $x_n^k \to x_n$ for all $n \in \mathbb{N}$.

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Proposition VII.1.18 (continued 2)

Proof (continued). To avoid a notational "Pandora's Box" (Conway, page 147) we denote this subsequence as $\{x_1^k \mid k \in \mathbb{N}_1\}$ where $\mathbb{N}_1$ is the appropriate infinite subset of $\mathbb{N}$. Consider the sequence of second entries of the sequence $\{\xi^k \mid k \in \mathbb{N}_1\}$. Then by the sequential compactness there is $x_2 \in X_2$ and infinite subset $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $\{x_2^k \mid k \in \mathbb{N}_2\} \to x_2$. Notice that since $\{x_1^k \mid k \in \mathbb{N}_1\}$ is convergent we still have $\{x_1^k \mid k \in \mathbb{N}_2\} \to x_1$. Continuing this process gives a decreasing sequence of infinite subsets of $\mathbb{N}$, $\mathbb{N}_1 \supset \mathbb{N}_2 \supset \cdots$ and points $x_n \in X_n$ such that

$$\{x_n^k \mid k \in \mathbb{N}_n\} \to x_n. \qquad (*)$$

Proposition VII.1.18 (continued 2)

Proof (continued). To avoid a notational "Pandora's Box" (Conway, page 147) we denote this subsequence as $\{x_1^k \mid k \in \mathbb{N}_1\}$ where $\mathbb{N}_1$ is the appropriate infinite subset of $\mathbb{N}$. Consider the sequence of second entries of the sequence $\{\xi^k \mid k \in \mathbb{N}_1\}$. Then by the sequential compactness there is $x_2 \in X_2$ and infinite subset $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $\{x_2^k \mid k \in \mathbb{N}_2\} \to x_2$. Notice that since $\{x_1^k \mid k \in \mathbb{N}_1\}$ is convergent we still have $\{x_1^k \mid k \in \mathbb{N}_2\} \to x_1$. Continuing this process gives a decreasing sequence of infinite subsets of $\mathbb{N}$, $\mathbb{N}_1 \supset \mathbb{N}_2 \supset \cdots$ and points $x_n \in X_n$ such that

$$\{x_n^k \mid k \in \mathbb{N}_n\} \to x_n. \qquad (*)$$

Let k_j be the jth integer in $\mathbb{N}_j$ and consider $\{\xi^{k_j}\}$. We claim $\{\xi^{k_j}\} \to \xi = \{x_n\}$ as $j \to \infty$. To show this, by Claim 3, it suffices to show that $x_n = \lim_{k_j \to \infty} x_n^{k_j}$ for all $n \in \mathbb{N}$. But since $\mathbb{N}_j \subset \mathbb{N}_n$ for $j \geq n$ then $\{x_n^{k_j} \mid j \geq n\}$ is a subsequence of $\{x_n^k \mid k\ in \mathbb{N}_n\}$. So $(*)$ then implies $\{x_n^{k_j} \mid j \geq n\} \to x_n$, or $x_n = \lim_{k_j \to \infty} x_n^{k_j}$.

Proposition VII.1.18 (continued 2)

Proof (continued). To avoid a notational "Pandora's Box" (Conway, page 147) we denote this subsequence as $\{x_1^k \mid k \in \mathbb{N}_1\}$ where $\mathbb{N}_1$ is the appropriate infinite subset of $\mathbb{N}$. Consider the sequence of second entries of the sequence $\{\xi^k \mid k \in \mathbb{N}_1\}$. Then by the sequential compactness there is $x_2 \in X_2$ and infinite subset $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $\{x_2^k \mid k \in \mathbb{N}_2\} \to x_2$. Notice that since $\{x_1^k \mid k \in \mathbb{N}_1\}$ is convergent we still have $\{x_1^k \mid k \in \mathbb{N}_2\} \to x_1$. Continuing this process gives a decreasing sequence of infinite subsets of $\mathbb{N}$, $\mathbb{N}_1 \supset \mathbb{N}_2 \supset \cdots$ and points $x_n \in X_n$ such that

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Proposition VII.1.18 (continued 3)

Proposition VII.1.18. The space $(\prod_{n=1}^{\infty} X_n, d)$ of the previous definition is a metric space. If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ if and only if $x_n^k \to x - n$ for all $n \in \mathbb{N}$. also, if each (X_n, d) is compact then X is compact.

Proof (continued). Therefore $\{\xi^{k_j}\} \to \xi = \{x_n\}$ and $\{\xi^{k_j}\}$ is a convergent subsequence of $\{\xi^k\}$. Therefore (X, d) is sequentially compact and, by Lemma II.4.9, compact. □

Proposition VII.1.18 (continued 3)

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Proposition VII.1.22

Proposition VII.1.22. Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Then $\mathcal{F}$ is equicontinuous over each compact subset of G.

Proof. Let $K \subset G$ be compact and fix $\varepsilon > 0$. Then by the definition of equicontinuous, for each $w \in K$ there is a $\delta_w > 0$ such that

$$|w - w'| < \delta_w \text{ implies } d(f(w'), f(w)) < \varepsilon/2 \qquad (*)$$

for all $f \in \mathcal{F}$.

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for all $f \in \mathcal{F}$. Now $\{B(w; \delta_w) \mid w \in K\}$ is an open cover of K. Since K is compact, it is sequentially compact (Theorem II.4.9) and so by the Lebesgue Covering Lemma (Lemma II.4.8) there is $\delta > 0$ such that for each $z \in K$, $B(z, \delta)$ is contained in one of the sets of this cover.

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Proposition VII.1.22 (continued)

Proposition VII.1.22. Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of G. Then $\mathcal{F}$ is equicontinuous over each compact subset of G.

Proof (continued). So by the Triangle Inequality

$$d(f(z), f(z')) \leq d(f(z), f(w)) + d(f(w), f(z')) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $|z - z'| < \delta$ and for all $f \in \mathcal{F}$. So $\mathcal{F}$ is equicontinuous over K. □

Theorem VII.1.23.

Theorem VII.1.23. Arzela-Ascoli Theorem
A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if the following two conditions are satisfied:

(a) For each $z \in G$, we have that $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in Ω;

(b) $\mathcal{F}$ is equicontinuous at each point of G.

Proof. (1) Suppose that $\mathcal{F}$ is normal. Notice that for each $z \in G$, the map of $C(G, \Omega)$ given by $f \to f(z)$ is continuous: Let $\varepsilon > 0$ and let $K = \{z\}$ be a compact subset of G.

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Theorem VII.1.23 (continued 1).

Proof (continued). Since $\mathcal{F}^-$ is compact by Proposition VII.1.15, its image is compact in Ω under the mapping described. Now if g is a limit point of the image of $\mathcal{F}$, $\{f(z) \mid f \in \mathcal{F}\}$, since the mapping is continuous. So the image of $\mathcal{F}^-$ is a subset of $\{f(z) \mid f \in \mathcal{F}\}^-$. The only points not in the image of $\mathcal{F}$ would have to be limit points of $\{f(z) \mid f \in \mathcal{F}\}^-$. But a limit point of $\{f(z) \mid f \in \mathcal{F}\}^-$. is also a limit point of $\mathcal{F}^-$ and since $\mathcal{F}^-$ is closed, then $\mathcal{F}^-$ must also contain such points. Hence the image of $\mathcal{F}^- = \{f(z) \mid f \in \mathcal{F}\}^-$ an d$\{f(z) \mid f \in \mathcal{F}\}$ has a compact closure. This gives (a).

Theorem VII.1.23 (continued 1).

Proof (continued). Since $\mathcal{F}^-$ is compact by Proposition VII.1.15, its image is compact in Ω under the mapping described. Now if g is a limit point of the image of $\mathcal{F}$, $\{f(z) \mid f \in \mathcal{F}\}$, since the mapping is continuous. So the image of $\mathcal{F}^-$ is a subset of $\{f(z) \mid f \in \mathcal{F}\}^-$. The only points not in the image of $\mathcal{F}$ would have to be limit points of $\{f(z) \mid f \in \mathcal{F}\}^-$. But a limit point of $\{f(z) \mid f \in \mathcal{F}\}^-$. is also a limit point of $\mathcal{F}^-$ and since $\mathcal{F}^-$ is closed, then $\mathcal{F}^-$ must also contain such points. Hence the image of $\mathcal{F}^- = \{f(z) \mid f \in \mathcal{F}\}^-$ an d$\{f(z) \mid f \in \mathcal{F}\}$ has a compact closure. This gives (a).

Now for (b). Fix a point $z_0 \in G$ and let $\varepsilon > 0$. Choose $R > 0$ so that $K = \overline{B}(z_o; R) \subset G$. Then K is compact (by Heine-Borel; remember, $G \subset \mathbb{C}$).

Theorem VII.1.23 (continued 1).

Proof (continued). Since $\mathcal{F}^-$ is compact by Proposition VII.1.15, its image is compact in Ω under the mapping described. Now if g is a limit point of the image of $\mathcal{F}$, $\{f(z) \mid f \in \mathcal{F}\}$, since the mapping is continuous. So the image of $\mathcal{F}^-$ is a subset of $\{f(z) \mid f \in \mathcal{F}\}^-$. The only points not in the image of $\mathcal{F}$ would have to be limit points of $\{f(z) \mid f \in \mathcal{F}\}^-$. But a limit point of $\{f(z) \mid f \in \mathcal{F}\}^-$. is also a limit point of $\mathcal{F}^-$ and since $\mathcal{F}^-$ is closed, then $\mathcal{F}^-$ must also contain such points. Hence the image of $\mathcal{F}^- = \{f(z) \mid f \in \mathcal{F}\}^-$ an d$\{f(z) \mid f \in \mathcal{F}\}$ has a compact closure. This gives (a).

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$$\sup\{f(f(z), f_k(z)) \mid z \in K\} < \varepsilon/3. \qquad (1.24)$$

Theorem VII.1.23 (continued 1).

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Theorem VII.1.23 (continued 2).

Proof (continued). But since each $f_k \in C(G, \Omega)$ is continuous, then there is a δ, $0 < \delta < R$, such that $|z - z_0| < \delta$ implies that $d(f_k(z), f_k(z_0)) < \varepsilon/3$ for all $1 \leq k \leq n$. Therefore if $|z - z_0| < \delta$, $f \in \mathcal{F}$ an dk is chosen so that (1.24) holds, then by the Triangle Inequality

$$d(f(z), f(z_0)) \leq d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So $\mathcal{F}$ is equicontinuous at the point z_0. Since z_0 is arbitrary, $\mathcal{F}$ is equicontinuous at each point of G and (b) holds.

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(2) Now suppose $\mathcal{F}$ satisfies conditions (a) and (b). Let $\{z_n\}$ be an enumeration of all the points in G with rational real and imaginary parts (so the points form a sequence). Then for any $z \in G$ an d$\delta > 0$ there is $z_n \in \{z_n\}$ with $|z - z_n| < \delta$. For each $n \ge 1$ let $X_n = \{f(z_n) \mid f \in \mathcal{F}\}^- \subset \Omega$. From hypothesis (a), (X_n, d) is a compact metric space.

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Theorem VII.1.23 (continued 3).

Proof (continued). So by Proposition VII.1.18 Claim 1, $X = \prod_{n-=1}^{\infty} X_n$ is a compact metric space. Fro $f \in \mathcal{F}$ define $\tilde{f} \in X$ by $f = \{f(z_1), f(z_2), \ldots\}$. Let $\{f_k\}$ be a sequence in $\mathcal{F}$. We will show that $\{f_k\}$ has a convergent subsequence and hence conclude that $\mathcal{F}$ is normal (actually, we will conclude that $\{f_k\}$ itself is convergent, so under (a) and (b), every sequence in $\mathcal{F}$ is convergent!). So $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. Thus, by Corollary II.4.6, there is $\xi \in X$ and a subsequence of $\{f_k\}$ which converges to ξ.

Theorem VII.1.23 (continued 3).

Proof (continued). So by Proposition VII.1.18 Claim 1, $X = \prod_{n-=1}^{\infty} X_n$ is a compact metric space. Fro $f \in \mathcal{F}$ define $\tilde{f} \in X$ by $f = \{f(z_1), f(z_2), \ldots\}$. Let $\{f_k\}$ be a sequence in $\mathcal{F}$. We will show that $\{f_k\}$ has a convergent subsequence and hence conclude that $\mathcal{F}$ is normal (actually, we will conclude that $\{f_k\}$ itself is convergent, so under (a) and (b), every sequence in $\mathcal{F}$ is convergent!). So $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. Thus, by Corollary II.4.6, there is $\xi \in X$ and a subsequence of $\{f_k\}$ which converges to ξ. "For the sake of convenient notation" we eliminate the layer of subscripts denoting this subsequence of $\{\tilde{f}_k\}$ simply as $\{\tilde{f}_k\}$ and so we notationally have $\xi = \lim \tilde{f}_k$. Thus, by Proposition VII.1.18 Claim 2,

$$\lim_{k \to \infty} f_k(z_n) = \omega_n \qquad (1.25)$$

where $\xi = \{\omega_n\}$. We'll show sequence $\{f_n\}$ is Cauchy in $C(G, \Omega)$ and, since $C(G, \Omega)$ is complete by Proposition VII.1.12, the existence of f is guaranteed.

Theorem VII.1.23 (continued 3).

Proof (continued). So by Proposition VII.1.18 Claim 1, $X = \prod_{n-=1}^{\infty} X_n$ is a compact metric space. Fro $f \in \mathcal{F}$ define $\tilde{f} \in X$ by $f = \{f(z_1), f(z_2), \ldots\}$. Let $\{f_k\}$ be a sequence in $\mathcal{F}$. We will show that $\{f_k\}$ has a convergent subsequence and hence conclude that $\mathcal{F}$ is normal (actually, we will conclude that $\{f_k\}$ itself is convergent, so under (a) and (b), every sequence in $\mathcal{F}$ is convergent!). So $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. Thus, by Corollary II.4.6, there is $\xi \in X$ and a subsequence of $\{f_k\}$ which converges to ξ. "For the sake of convenient notation" we eliminate the layer of subscripts denoting this subsequence of $\{\tilde{f}_k\}$ simply as $\{\tilde{f}_k\}$ and so we notationally have $\xi = \lim \tilde{f}_k$. Thus, by Proposition VII.1.18 Claim 2,

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Theorem VII.1.23 (continued 4).

Proof (continued). Let K be a compact subset of G and let $\varepsilon > 0$. By Lemma VII.1.10(b) it suffices to show that $\{f_n\}$ converges to f on K uniformly. That is, we need $J \in \mathbb{N}$ such that for $j, k \geq J$ we have

$$\sup\{d(f_k(z), f_j(z)) \mid z \in K\} < \varepsilon \qquad (1.26)$$

(here we are using "uniformly Cauchy" but the Cauchyness implies [pointwise] convergence and then uniform convergence). Since K is compact an d$\partial G = G^- \cap (\mathbb{C} \setminus G)^-$ is closed, then by Theorem II.5.17, $R = d(K, \partial G) > 0$. Let $K_1 = \{z \mid d(z, K) \leq R/2\}$. Since K is a compact subset of $\mathbb{C}$ then K_1 is compact and $K \subset \text{int}(K_1) \subset K_1 \subset G$. Since $\mathcal{F}$ is equicontinuous at each point of G by hypothesis (b) and so $\mathcal{F}$ is equicontinuous on K_1 by Proposition VII.1.22.

Theorem VII.1.23 (continued 4).

Proof (continued). Let K be a compact subset of G and let $\varepsilon > 0$. By Lemma VII.1.10(b) it suffices to show that $\{f_n\}$ converges to f on K uniformly. That is, we need $J \in \mathbb{N}$ such that for $j, k \geq J$ we have

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$$d(f(z), f(z')) < \varepsilon/3 \text{ for all } f \in \mathcal{F}. \qquad (1.27)$$

Theorem VII.1.23 (continued 4).

Proof (continued). Let K be a compact subset of G and let $\varepsilon > 0$. By Lemma VII.1.10(b) it suffices to show that $\{f_n\}$ converges to f on K uniformly. That is, we need $J \in \mathbb{N}$ such that for $j, k \geq J$ we have

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Theorem VII.1.23 (continued 5).

Proof (continued). Now let $D = \{z_n\} \cap K_1$. If $z \in K$ then there is a z_n with $|z - z_n| < \delta$ since $\{z_n\}$ is dense in G. But $\delta < R/2$ implies that $d(z_n, K) < R/2$, and so $z_n \in K_1$ (by the definition of K_1), or $z_n \in D$ by the definition of D. So every element z of K is within δ of an element of D. Hence $\{B(w;\delta) \mid w \in D\}$ is an open cover of K. Since K is compact, there are $w_1, w_2, \ldots, w_n \in D$ such that $K \subset \cup_{i=1}^{n} B(w_i;\delta)$. Since $\lim_{k\to\infty} f_k(w_j)$ exists for $1 \le i \le n$ by (1.25), there is $J \in \mathbb{N}$ such that for $k, j \ge J$ we have

$$d(f_k(w_i), f_j(w_i)) < \varepsilon/3 \text{ for } i = 1, 2, \ldots, n. \qquad (1.28)$$

Theorem VII.1.23 (continued 5).

Proof (continued). Now let $D = \{z_n\} \cap K_1$. If $z \in K$ then there is a z_n with $|z - z_n| < \delta$ since $\{z_n\}$ is dense in G. But $\delta < R/2$ implies that $d(z_n, K) < R/2$, and so $z_n \in K_1$ (by the definition of K_1), or $z_n \in D$ by the definition of D. So every element z of K is within δ of an element of D. Hence $\{B(w; \delta) \mid w \in D\}$ is an open cover of K. Since K is compact, there are $w_1, w_2, \ldots, w_n \in D$ such that $K \subset \cup_{i=1}^{n} B(w_i; \delta)$. Since $\lim_{k\to\infty} f_k(w_j)$ exists for $1 \le i \le n$ by (1.25), there is $J \in \mathbb{N}$ such that for $k, j \ge J$ we have

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Let $z \in K$ be arbitrary and let w_i be such that $|w_i - z| < \delta$ (remember $B(w_i; \delta)$, $i = 1, 2, \ldots, n$ is a covering of K).

Theorem VII.1.23 (continued 5).

Proof (continued). Now let $D = \{z_n\} \cap K_1$. If $z \in K$ then there is a z_n with $|z - z_n| < \delta$ since $\{z_n\}$ is dense in G. But $\delta < R/2$ implies that $d(z_n, K) < R/2$, and so $z_n \in K_1$ (by the definition of K_1), or $z_n \in D$ by the definition of D. So every element z of K is within δ of an element of D. Hence $\{B(w;\delta) \mid w \in D\}$ is an open cover of K. Since K is compact, there are $w_1, w_2, \ldots, w_n \in D$ such that $K \subset \cup_{i=1}^n B(w_i;\delta)$. Since $\lim_{k\to\infty} f_k(w_j)$ exists for $1 \le i \le n$ by (1.25), there is $J \in \mathbb{N}$ such that for $k, j \ge J$ we have

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Theorem VII.1.23 (continued 6).

Proof (continued). For $j, k \geq J$ we have from (1.27) and (1.28) that

$$d(f_k(z), f_j(z)) \leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

(the first inequality follows from (1.27) since (1.27) holds for all $f \in \mathcal{F}$ and $f_k \in \mathcal{F}$; the second inequality holds from (1.28); the third inequality follows as the first). Since z was arbitrary, we have (1.26): for $j, k \geq J$, $\sup\{d(f_k(z), f_j(z)) \mid z \in K\} < \varepsilon$ (since K is compact, the "sup" can be replaced with "max" and the strict inequality remains). So $\{f_k\}$ is uniformly continuous on K and by Lemma VII.1.10(b), $\{f_k\}$ converges. □

Theorem VII.1.23 (continued 6).

Proof (continued). For $j, k \geq J$ we have from (1.27) and (1.28) that

$$d(f_k(z), f_j(z)) \leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

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