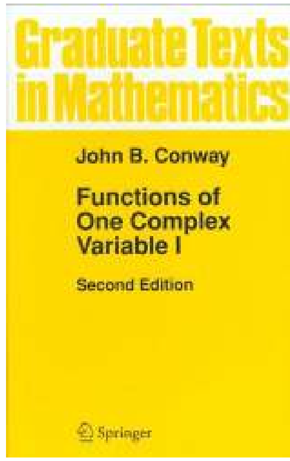
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.2. Spaces of Analytic Functions—Proofs of Theorems



Theorem VII.1.2

Theorem VII.2.1. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G,\mathbb{C})$ such that $\lim_{n\to\infty} f_n = f$, then f is analytic and the derivatives satisfy $\lim_{n\to\infty} f_n^{(k)} = f^{(k)}$ for each $k \in \mathbb{N}$.

Proof. First, we show that f is analytic using Morera's Theorem (Theorem IV.5.10). Let T be a triangle contained inside a disk $D \subset G$. Since T is a compact set and $\{f_n\} \to f$ by hypothesis, then $\{f_n\}$ converges to f uniformly on T by Proposition VII.1.10(b). Since each f_n is analytic and T is closed, $\int_T f_n = 0$ by Cauchy's Theorem–Second Version (Theorem VI.6.6). Since the convergence on T is uniform then by Lemma IV.2.7

$$0 = \lim_{n\to\infty}\left(\int_T f_n\right) = \int_T \left(\lim_{n\to\infty} f_n\right) = \int_T f.$$

So by Morera's Theorem, f is analytic in every disk $D \subset G$ and so f is analytic in G.

Theorem VII.1.2 (continued 1)

Theorem VII.2.1. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G,\mathbb{C})$ such that $\lim_{n\to\infty} f_n = f$, then f is analytic and the derivatives satisfy $\lim_{n\to\infty} f_n^{(k)} = f^{(k)}$ for each $k \in \mathbb{N}$.

Proof (continued). Now for the derivatives. Let $D = \overline{B}(a;r) \subset G$. Then there is $R > r$ such that $\overline{B}(a;R) \subset G$. If γ is the circle $z = a + Re^{it}$, $t \in [0, 2\pi]$, then by Cauchy's Integral Formula (actually, Corollary IV.5.9) for $z \in D$ we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^{k+1}}\,dw,\quad f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,dw,$$

$$\text{and so } f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw.$$

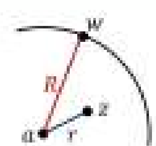
Let $M_n = \max\{|f_n(w) - f(w)| \mid |w-a| = R\}$.

Theorem VII.1.2 (continued 2)

Proof (continued). By Proposition IV.1.17(b),

$$\begin{aligned}
|f_n^{(k)}(z) - f^{(k)}(z)| &= \frac{k!}{2\pi}\left|\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw\right| \\
&\le \frac{k!}{2\pi}\int_\gamma \frac{|f_n(w)-f(w)|}{|w-z|^{k+1}}\,|dw| \\
&\le \frac{k!}{2\pi}\frac{M_n 2\pi R}{(R-r)^{k+1}} \text{ for } |z-a| \le r\,* \\
&= \frac{M_n k! R}{(R-r)^{k+1}}. \qquad (2.2)
\end{aligned}$$

*With $|z-a| \le r$ and $|w-a| = R$ we have $|w-z| \ge R-r$, or $1/|w-z| \le 1/(R-r)$:



Theorem VII.1.2 (continued 3)

Theorem VII.2.1. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G,\mathbb{C})$ such that $\lim_{n\to\infty} f_n = f$, then f is analytic and the derivatives satisfy $\lim_{n\to\infty} f_n^{(k)} = f^{(k)}$ for each $k \in \mathbb{N}$.

Proof (continued). Since $f_n \to f$ in $C(G,\mathbb{C})$, then by Proposition VII.1.10(b), $f_n \to f$ uniformly on compact set $\overline{B}(a;R)$ and so $\lim M_n = 0$. So by (2.2), we have that $f_n^{(k)} \to f^{(k)}$ uniformly on $\overline{B}(a;r) \subset \overline{B}(a;R)$. Now if K is an arbitrary compact (closed and bounded) subset of $G \subset \mathbb{C}$ and $0 < r < d(K,\partial G)$ then there are (finitely many) $a_1, a_2, \ldots, a_n \in K$ such that $K \subset \cup_{j=1}^n B(a_j;r)$ since K is compact. Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j;r) \subset \overline{B}(a_j;r)$ for $j = 1,2,\ldots,n$, then the convergence is uniform on K. Since uniform convergence implies convergence with respect to ρ as commented above, then $f_n^{(k)} \to f^{(k)}$ is $C(G,\mathbb{C})$ for each $k \geq 1$. □

Theorem VII.2.5

Theorem VII.2.5. Hurwitz's Theorem. Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a;R) \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a;R)$.

Proof. Since $f(z) \neq 0$ for $|z-z| = R$, then

$$\delta = \inf\{|f(z)| \mid |z-a| = R\} > 0$$

since this is the distance between the compact set of real numbers $\{|f(z)| \mid |z-a| = R\}$ and the closed set $\{0\} \subset \mathbb{R}$. By Theorem II.5.17. Since $|z-a| = R$ is a compact set, then by Proposition VII.1.10(b) $f_n \to f$ uniformly on $|z-a| = R$ so there is $N \in \mathbb{N}$ such that if $n \geq N$ and $|z-a| = R$ then $f_n(z) \neq 0$, or $|f(z) - f_n(z)| < \delta/2 < |f(z)| + |f_n(z)|$. So by Rouche's Theorem (Theorem V.3.8), f and f_n have the same number of zeros in $B(a;R)$. □

Theorem VII.2.9. Montel's Theorem

Theorem VII.2.9. Montel's Theorem. A family $\mathcal{F} \subset H(G)$ if normal if and only if $\mathcal{F}$ is locally bounded.

Proof. Let $\mathcal{F}$ be normal. ASSUME $\mathcal{F}$ is not locally bounded. By Lemma VII.2.8, there is a compact set $K \subset G$ such that $\sup\{|f(z)| \mid z \in K, f \in \mathcal{F}\} = \infty$. That is, there is a sequence $\{f_n\} \subset \mathcal{F}$ such that $\sup\{|f_n(z)| \mid z \in z \in K\} = \infty$. That is, there is a sequence $\{f_n\} \subset \mathcal{F}$ such that $\sup\{|f_n(z)| \mid z \in K\} \geq n$. Since $\mathcal{F}$ is normal then (by definition of normal) there is $f \in H(G)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ (with respect to the metric ρ on $C(G,\mathbb{C})$ and so by Theorem VII.1.10(b), $f_{n_k} \to f$ uniformly on K. Therefore $\sup\{|f_{n_k}(z) - f(z)| \mid z \in K\} \to 0$ as $k \to \infty$. Since f is continuous and K is compact then there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in K$. Now for all $z \in K$ we have

$$|f_{n_k}(z)| = |f_{n_k}(z) - f(z) + f(z)| \leq |f_{n_k}(z) - f(z)| + |f(z)| \leq |f_{n_k}(z) - f(z)| + M,$$

...

Theorem VII.2.9. Montel's Theorem (continued 1)

Proof (continued). . . . and so

$$n_k \leq \sup\{|f_{n_k}(z)| \mid z \in K\} \leq \sup\{|f_{n_k}(z) - f(z)| \mid z \in K\} + M. \qquad (*)$$

Now $n_k \to \infty$ as $k \to \infty$, but this implies that the limit as $k \to \infty$ in the left had side of $(*)$is ∞ while the limit of the right hand side is M, a CONTRADICTION. So the assumption is false and f is locally bounded.

Let $\mathcal{F}$ be locally bounded. We will use the Arzela-Ascoli Theorem (Theorem VII.1.23) to prove that $\mathcal{F}$ is normal. Let $a \in G$. Then the local boundedness of $\mathcal{F}$ implies that for some $M \geq 0$ and some $r \geq 0$ we have $|f(z)| \leq M$ for $|z-a| < r$ and for all $f \in \mathcal{F}$. So, in particular, $\{f(z) \mid f \in \mathcal{F}\}$ is bounded and so has compact closure (by Heine-Borel). That is, part (a) of the Arzela-Ascoli Theorem is satisfied. Again, fix $a \in G$ and let $\varepsilon > 0$. The local boundedness of $\mathcal{F}$ implies that there is $r > 0$ and $M > 0$ such that $\overline{B}(a;r) \subset G$ and $|f(z)| \leq M$ for all $z \in \overline{B}(a;r)$ and for all $f \in \mathcal{F}$ (technically, we should have $B(a;r) \subset G$, but we can still draw our conclusion by adjusting r to $r/2$, say).

Theorem VII.2.9. Montel's Theorem (continued 1)

Proof (continued). Let $|z-a| < r/2$ and $f \in \mathcal{F}$. Set $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. By Cauchy's Integral Formula (first version, Theorem IV.5.4) applied to $f(a)$ and $f(z)$ we have:

$$\begin{aligned}
|f(a) - f(z)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\,dw - \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dw \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{f(w)(w-z) - f(w)(w-a)}{(w-a)(w-z)}\,dw \right| \\
&= \frac{1}{2\pi} \int_\gamma \left| \frac{f(w)(a-z)}{(w-a)(w-z)} \right| |dw| \\
&\leq \frac{M|a-z|}{2\pi} \int_\gamma \frac{|dw|}{|w-a||w-z|} \\
&\leq \frac{M|a-z|}{2\pi} \frac{2\pi r}{r(r/2)} \text{ since } |w-z| = |w-a+a-z| \\
&\qquad \geq |w-a| - |a-z| = r - \frac{r}{2} = \frac{r}{2} \text{ because } |z-a| < \frac{r}{2}
\end{aligned}$$

Theorem VII.2.9. Montel's Theorem (continued 2)

Proof (continued). ...and so

$$|f(a) - f(z)| \leq \frac{2M}{r}|a-z|.$$

Let $\delta < \min\{r/2, r\varepsilon/(4M)\}$. Then $|a-z| < \delta$ implies

$$\begin{aligned}
|f(a) - f(z)| &\leq 2M|a-z|/r \text{ since } |a-z| < r/2 \\
&< (2M/r)(r\varepsilon/(4M)) \text{ since } |a-z| < r\varepsilon/(4M) \\
&= \varepsilon/2 < \varepsilon.
\end{aligned}$$

Since $f \in \mathcal{F}$ was arbitrary, we have that $\mathcal{F}$ is equicontinuous at point a. Since a is an arbitrary point of G, then part (b) of the Arzela-Ascoli Theorem is satisfied. The Arzela-Ascoli Theorem then implies that $\mathcal{F}$ is normal, as claimed. □

Corollary VII.2.10

Corollary VII.2.10. A set $\mathcal{F} \subset H(G)$ is compact if and only if it is closed and locally bounded.

Proof. Let $\mathcal{F} \subset H(G)$ be closed and locally bounded. By Montel's Theorem (Theorem VII.2.9), $\mathcal{F}$ is normal. So for any sequence $\{f_n\} \subset \mathcal{F}$, there is (by the definition of normal) a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges in $H(G)$ to, say, f. Since $\mathcal{F}$ is closed, the first limit function $f \in \mathcal{F}$. So $\mathcal{F}$ is sequentially compact. By Theorem II.4.9, $\mathcal{F}$ is compact.

Let $\mathcal{F}$ be compact. Then $\mathcal{F}$ is closed by Theorem II.4.3(a). So $\mathcal{F}$ has compact closure and so by Proposition VII.1.15 $\mathcal{F}$ is normal. By Montel's Theorem, $\mathcal{F}$ is locally bounded. □