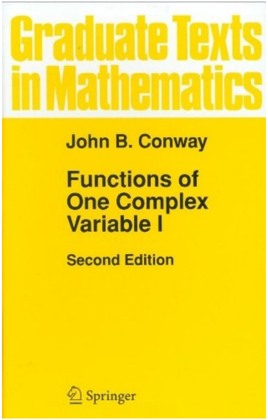
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.2. Spaces of Analytic Functions—Proofs of Theorems



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Theorem VII.1.2

Theorem VII.2.1. If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G, \mathbb{C})$ such that $\lim_{n\to\infty} f_n = f$, then f is analytic and the derivatives satisfy $\lim_{n\to\infty} f_n^{(k)} = f^{(k)}$ for each $k \in \mathbb{N}$.

Proof. First, we show that f is analytic using Morera's Theorem (Theorem IV.5.10). Let T be a triangle contained inside a disk $D \subset G$. Since T is a compact set and $\{f_n\} \to f$ by hypothesis, then $\{f_n\}$ converges to f uniformly on T by Proposition VII.1.10(b).

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$$0 = \lim_{n\to\infty} \left(\int_T f_n \right) = \int_T \left(\lim_{n\to\infty} f_n \right) = \int_T f.$$

So by Morera's Theorem, f is analytic in every disk $D \subset G$ and so f is analytic in G.

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Proof (continued). Now for the derivatives. Let $D = \overline{B}(a; r) \subset G$. Then there is $R > r$ such that $\overline{B}(a; R) \subset G$. If γ is the circle $z = a + Re^{it}$, $t \in [0, 2\pi]$, then by Cauchy's Integral Formula (actually, Corollary IV.5.9) for $z \in D$ we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^{k+1}}\,dw,\quad f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,dw,$$

$$\text{and so } f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\,dw.$$

Let $M_n = \max\{|f_n(w) - f(w)| \mid |w - a| = R\}$.

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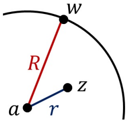
Theorem VII.1.2 (continued 2)

Proof (continued). By Proposition IV.1.17(b),

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &= \frac{k!}{2\pi}\left|\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw\right| \\ &\le \frac{k!}{2\pi}\int_\gamma \frac{|f_n(w)-f(w)|}{|w-z|^{k+1}}\,|dw| \\ &\le \frac{k!}{2\pi}\frac{M_n 2\pi R}{(R-r)^{k+1}} \text{ for } |z-a|\le r\,* \\ &= \frac{M_n k! R}{(R-r)^{k+1}}. \qquad (2.2) \end{aligned}$$

*With $|z-a| \le r$ and $|w-a| = R$ we have $|w-z| \ge R-r$, or $1/|w-z| \le 1/(R-r)$:



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Proof (continued). Since $f_n \to f$ in $C(G, \mathbb{C})$, then by Proposition VII.1.10(b), $f_n \to f$ uniformly on compact set $\overline{B}(a; R)$ and so $\lim M_n = 0$. So by (2.2), we have that $f_n^{(k)} \to f^{(k)}$ uniformly on $\overline{B}(a; r) \subset \overline{B}(a; R)$. Now if K is an arbitrary compact (closed and bounded) subset of $G \subset \mathbb{C}$ and $0 < r < d(K, \partial G)$ then there are (finitely many) $a_1, a_2, \ldots, a_n \in K$ such that $K \subset \cup_{j=1}^n B(a_j; r)$ since K is compact. Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j; r) \subset \overline{B}(a_j; r)$ for $j = 1, 2, \ldots, n$, then the convergence is uniform on K.

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Theorem VII.2.5

Theorem VII.2.5. Hurwitz's Theorem. Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a; R) \subset G$, and $f(z) \neq 0$ for $|z - a| = R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a; R)$.

Proof. Since $f(z) \neq 0$ for $|z - z| = R$, then

$$\delta = \inf\{|f(z)| \mid |z - a| = R\} > 0$$

since this is the distance between the compact set of real numbers $\{|f(z)| \mid |z - a| = R\}$ and the closed set $\{0\} \subset \mathbb{R}$. By Theorem II.5.17.

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$$|f_{n_k}(z)| = |f_{n_k}(z) - f(z) + f(z)| \leq |f_{n_k}(z) - f(z)| + |f(z)| \leq |f_{n_k}(z) - f(z)| + M,$$

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Theorem VII.2.9. Montel's Theorem (continued 1)

Proof (continued). ... and so

$$n_k \leq \sup\{|f_{n_k}(z)| \mid z \in K\} \leq \sup\{|f_{n_k}(z) - f(z)| \mid z \in K\} + M. \qquad (*)$$

Now $n_k \to \infty$ as $k \to \infty$, but this implies that the limit as $k \to \infty$ in the left had side of $(*)$is ∞ while the limit of the right hand side is M, a CONTRADICTION. So the assumption is false and f is locally bounded.

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Theorem VII.2.9. Montel's Theorem (continued 1)

Proof (continued). Let $|z-a| < r/2$ and $f \in \mathcal{F}$. Set $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. By Cauchy's Integral Formula (first version, Theorem IV.5.4) applied to $f(a)$ and $f(z)$ we have:

$$\begin{aligned}
|f(a) - f(z)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw - \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{f(w)(w-z) - f(w)(w-a)}{(w-a)(w-z)}\, dw \right| \\
&= \frac{1}{2\pi} \int_\gamma \left| \frac{f(w)(a-z)}{(w-a)(w-z)} \right| |dw| \\
&\le \frac{M|a-z|}{2\pi} \int_\gamma \frac{|dw|}{|w-a||w-z|} \\
&\le \frac{M|a-z|}{2\pi} \frac{2\pi r}{r(r/2)} \text{ since } |w-z| = |w-a+a-z| \\
&\quad \ge |w-a| - |a-z| = r - \frac{r}{2} = \frac{r}{2} \text{ because } |z-a| < \frac{r}{2}
\end{aligned}$$

Theorem VII.2.9. Montel's Theorem (continued 1)

Proof (continued). Let $|z-a|<r/2$ and $f\in\mathcal{F}$. Set $\gamma(t)=a+re^{it}$, $t\in[0,2\pi]$. By Cauchy's Integral Formula (first version, Theorem IV.5.4) applied to $f(a)$ and $f(z)$ we have:

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|f(a)-f(z)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(w-z)-f(w)(w-a)}{(w-a)(w-z)}\,dw\right| \\
&= \frac{1}{2\pi}\int_\gamma \left|\frac{f(w)(a-z)}{(w-a)(w-z)}\right|\,|dw| \\
&\le \frac{M|a-z|}{2\pi}\int_\gamma \frac{|dw|}{|w-a||w-z|} \\
&\le \frac{M|a-z|}{2\pi}\,\frac{2\pi r}{r(r/2)} \text{ since } |w-z|=|w-a+a-z| \\
&\quad \ge |w-a|-|a-z| = r-\frac{r}{2}=\frac{r}{2} \text{ because } |z-a|<\frac{r}{2}
\end{aligned}$$

Theorem VII.2.9. Montel's Theorem (continued 2)

Proof (continued). ... and so

$$|f(a) - f(z)| \leq \frac{2M}{r}|a - z|.$$

Let $\delta < \min\{r/2, r\varepsilon/(4M)\}$. Then $|a - z| < \delta$ implies

$$\begin{aligned} |f(a) - f(z)| &\leq 2M|a - z|/r \text{ since } |a - z| < r/2 \\ &< (2M/r)(r\varepsilon/(4M)) \text{ since } |a - z| < r\varepsilon/(4M) \\ &= \varepsilon/2 < \varepsilon. \end{aligned}$$

Since $f \in \mathcal{F}$ was arbitrary, we have that $\mathcal{F}$ is equicontinuous at point a. Since a is an arbitrary point of G, then part (b) of the Arzela-Ascoli Theorem is satisfied. The Arzela-Ascoli Theorem then implies that $\mathcal{F}$ is normal, as claimed. □

Theorem VII.2.9. Montel's Theorem (continued 2)

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Corollary VII.2.10

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Proof. Let $\mathcal{F} \subset H(G)$ be closed and locally bounded. By Montel's Theorem (Theorem VII.2.9), $\mathcal{F}$ is normal. So for any sequence $\{f_n\} \subset \mathcal{F}$, there is (by the definition of normal) a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges in $H(G)$ to, say, f. Since $\mathcal{F}$ is closed, the first limit function $f \in \mathcal{F}$. So $\mathcal{F}$ is sequentially compact. By Theorem II.4.9, $\mathcal{F}$ is compact.

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