

## Theorem VII.3.4

**Theorem VII.3.4.** Let  $\{f_n\}$  be a sequence in  $M(G)$  and suppose  $f_n \rightarrow f$  in  $C(G, \mathbb{C}_\infty)$ . Then either  $f$  is meromorphic or  $f \equiv \infty$ . If each  $f_n$  is analytic then either  $f$  is analytic on  $f \equiv \infty$ .

**Proof. (I)** Let  $a \in G$  with  $f(a) \neq \infty$ . Set  $M = |f(a)|$ . By Proposition VII.3.3(a), there is  $\rho > 0$  such that  $B_\infty(f(a); \rho) \subset B(f(z); M)$ . Since  $f_n \rightarrow f$  then there is  $n_0 \in \mathbb{N}$  such that  $d(f_n(a), f(a)) < \rho/2$  for all  $n \geq n_0$  (since convergence in  $C(G, \mathbb{C}_\infty)$  implies convergence on compact subsets of  $G$  and so implies pointwise convergence since  $\{a\} \subset G \subset \mathbb{C}$  is compact). Now  $\{f_-, f_1, f_2, \dots\} \subset C(G, \mathbb{C}_\infty)$  is compact since any open cover must include an open set containing  $f$  and this open set contains all by finitely many of the other elements of the set. So  $\{f, f_1, f_2, \dots\}$  has compact closure and by Proposition VII.1.15 it is normal. By the Arzela-Ascoli Theorem (Theorem VII.1.23(b))  $\{f, f_1, f_2, \dots\}$  is equicontinuous. So by the definition of equicontinuity, there is  $r_1 > 0$  such that  $|z - a| < r_1$  implies  $d(f_n(z), f_n(z)) < \rho/2$ .

## Complex Analysis

## Chapter VII. Compactness and Convergence in the Space of Analytic Functions

## VII.3. Spaces of Meromorphic Functions—Proofs of Theorems



Theorem VII.3.4

## Theorem VII.3.4 (continued 1)

**Proof (continued).** So for  $|z - a| \leq r < r_1$  and  $n \geq n_0$  we have

$$d(f_n(z), f(a)) \leq d(f_n(z), f_n(a)) + d(f_n(a), f(a)) < \rho/2 + \rho/2 = \rho.$$

By the choice of  $\rho$ ,  $B_\infty(f(a); \rho) \subset B(f(a); M)$ , so  $d(f_n(z), f(a)) < \rho$  implies  $f_n(z) \in B_\infty(f(a); \rho) \subset B(f(a); M)$  and so  $|f_n(z) - f(a)| < M$ . So for  $z \in \overline{B}(a; r)$  and  $n \geq n_0$  we have

$$|f_n(z)| = |f_n(z) - f(a) + f(a)| \leq |f_n(z) - f(a)| + |f(a)| < M + M = 2M.$$

Since  $f_n(z) \rightarrow f(z)$ , then  $f(z) \leq 2M$  for all  $z \in \overline{B}(a; r)$ . So

$$\begin{aligned} d(f_n(z), f(z)) &= \frac{2|f_n(z) - f(z)|}{\{(1 + |f_n(z)|^2)(1 + |f(z)|^2)\}^{1/2}} \\ &\geq \frac{2|f_n(z) - f(z)|}{\{(1 + (2M)^2)(1 + (2M)^2)\}^{1/2}} = \frac{2|f_n(z) - f(z)|}{1 + 4M^2} \end{aligned}$$

for  $z \in \overline{B}(a; r)$  and  $n \geq n_0$ .

Theorem VII.3.4

## Theorem VII.3.4 (continued 2)

**Proof (continued).** By Proposition VII.1.10,  $f_n \rightarrow f$  in  $C(G, \mathbb{C}_\infty)$  implies that  $d(f_n(z), f(z)) \rightarrow 0$  uniformly on compact set  $\overline{B}(a; r)$ . So the above inequality implies that  $|f_n(z) - f(z)| \rightarrow 0$  uniformly in  $\overline{B}(a; r)$ . Now for  $n \geq n_0$ ,  $|f_n(z)| \leq 2M$  for all  $z \in \overline{B}(a; r)$  so for  $n \geq n_0$  each meromorphic  $f_n$  must be analytic on  $B(a; r)$  (since bounded  $f_n$  cannot have a pole in  $B(a; r)$ ). So  $\{f_n\}_{n \geq n_0}$  is a sequence of analytic functions which converges uniformly on  $\overline{B}(a; r)$  (and so converges uniformly on any compact subset of  $\overline{B}(a; r)$ ) and then by Proposition VII.1.10  $\{f_n\}_{n \geq n_0}$  converges in  $C(B(a; r), \mathbb{C}_\infty)$ . By Theorem VII.2.1, the limit function  $f$  is analytic on  $B(a; r)$ . That is, if  $f(a) \neq \infty$  then  $f$  is analytic in some neighborhood of  $a$ .

**(III)** Now suppose there is  $a \in G$  with  $f(a) = \infty$ . For  $g \in C(G, \mathbb{C}_\infty)$ , define  $\left(\frac{1}{g}\right) = \begin{cases} 1/g(z) & \text{if } g(z) \notin \{0, \infty\} \\ 0 & \text{if } g(z) = \infty \\ \infty & \text{if } g(z) = 0. \end{cases}$  Then  $1/g \in C(G, \mathbb{C}_\infty)$ .

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Theorem VII.3.4

## Theorem VII.3.4 (continued 1)

**Proof (continued).** So for  $|z - a| \leq r < r_1$  and  $n \geq n_0$  we have

$$d(f_n(z), f(a)) \leq d(f_n(z), f_n(a)) + d(f_n(a), f(a)) < \rho/2 + \rho/2 = \rho.$$

By the choice of  $\rho$ ,  $B_\infty(f(a); \rho) \subset B(f(a); M)$ , so  $d(f_n(z), f(a)) < \rho$  implies  $f_n(z) \in B_\infty(f(a); \rho) \subset B(f(a); M)$  and so  $|f_n(z) - f(a)| < M$ . So for  $z \in \overline{B}(a; r)$  and  $n \geq n_0$  we have

$$|f_n(z)| = |f_n(z) - f(a) + f(a)| \leq |f_n(z) - f(a)| + |f(a)| < M + M = 2M.$$

Since  $f_n(z) \rightarrow f(z)$ , then  $f(z) \leq 2M$  for all  $z \in \overline{B}(a; r)$ . So

$$\begin{aligned} d(f_n(z), f(z)) &= \frac{2|f_n(z) - f(z)|}{\{(1 + |f_n(z)|^2)(1 + |f(z)|^2)\}^{1/2}} \\ &\geq \frac{2|f_n(z) - f(z)|}{\{(1 + (2M)^2)(1 + (2M)^2)\}^{1/2}} = \frac{2|f_n(z) - f(z)|}{1 + 4M^2} \end{aligned}$$

for  $z \in \overline{B}(a; r)$  and  $n \geq n_0$ .

Theorem VII.3.4

## Theorem VII.3.4 (continued 2)

**Proof (continued).** By Proposition VII.1.10,  $f_n \rightarrow f$  in  $C(G, \mathbb{C}_\infty)$  implies that  $d(f_n(z), f(z)) \rightarrow 0$  uniformly on compact set  $\overline{B}(a; r)$ . So the above inequality implies that  $|f_n(z) - f(z)| \rightarrow 0$  uniformly in  $\overline{B}(a; r)$ . Now for  $n \geq n_0$ ,  $|f_n(z)| \leq 2M$  for all  $z \in \overline{B}(a; r)$  so for  $n \geq n_0$  each meromorphic  $f_n$  must be analytic on  $B(a; r)$  (since bounded  $f_n$  cannot have a pole in  $B(a; r)$ ). So  $\{f_n\}_{n \geq n_0}$  is a sequence of analytic functions which converges uniformly on  $\overline{B}(a; r)$  (and so converges uniformly on any compact subset of  $\overline{B}(a; r)$ ) and then by Proposition VII.1.10  $\{f_n\}_{n \geq n_0}$  converges in  $C(B(a; r), \mathbb{C}_\infty)$ . By Theorem VII.2.1, the limit function  $f$  is analytic on  $B(a; r)$ . That is, if  $f(a) \neq \infty$  then  $f$  is analytic in some neighborhood of  $a$ .

**(III)** Now suppose there is  $a \in G$  with  $f(a) = \infty$ . For  $g \in C(G, \mathbb{C}_\infty)$ , define  $\left(\frac{1}{g}\right) = \begin{cases} 1/g(z) & \text{if } g(z) \notin \{0, \infty\} \\ 0 & \text{if } g(z) = \infty \\ \infty & \text{if } g(z) = 0. \end{cases}$  Then  $1/g \in C(G, \mathbb{C}_\infty)$ .

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## Theorem VII.3.4 (continued 3)

**Proof (continued).** Since, as observed above,  $d(z_1, z_2) = d(1/z_1, 1/z_2)$  and  $d(z, 0) = d(1/z, \infty)$  for  $z \neq 0$ , then  $f_n \rightarrow f$  in  $C(G, \mathbb{C}_\infty)$  implies  $1/f_n \rightarrow 1/f$  in  $C(G, \mathbb{C}_\infty)$ . Since each  $f_n$  is meromorphic on  $G$ , then each  $1/f_n$  is meromorphic on  $G$ . Since  $(1/f)(z) = 0 \neq \infty$ , by Part I there is  $r > 0$  and  $n_0 \in \mathbb{N}$  such that  $1/f$  and  $1/f_n$  are analytic on  $B(a; r)$  for  $n \geq n_0$ , and  $1/f_n \rightarrow 1/f$  uniformly on  $B(a; r)$ . So by Proposition VII.1.10(b),  $1/f_n \rightarrow 1/f$  in  $H(G) \subset C(G, \mathbb{C})$ . So the hypotheses of Hurwitz's Theorem (Theorem VII.2.5) and the corollary to Hurwitz's Theorem, Corollary VII.2.6, are satisfied. Since  $1/f$  is analytic on  $B(a; r)$  and  $(1/f)(a) = 0$ , then by Corollary IV.3.10 then either  $1/f \equiv 0$  or there is  $R > 0$  such that  $B(a; R) \subset G$  and  $(1/f)(z) \neq 0$  for  $0 < |z - a| < R$ ; that is, either  $a/f \equiv 0$  or  $z = a$  is an isolated zero of  $1/f$ . Therefore, either  $f \equiv 0$  or  $f$  is meromorphic on  $B(a; R)$ . (Conway uses Hurwitz's Theorem to reach this conclusion.)

Combining Parts I and II, either  $f \equiv \infty$  or  $f$  is meromorphic in  $G$ ; that is,  $f \in M(G) \cup \{\infty\}$ .

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## Theorem VII.3.8

**Theorem VII.3.8.** A family  $\mathcal{F} \subset M(G)$  is normal in  $C(G, \mathbb{C}_\infty)$  if and only if  $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$  is locally bounded.

**Proof of locally bounded implies normal.** Let  $\mathcal{F} \subset M(G)$  with  $\mu(\mathcal{F})$  locally bounded. Notice that  $\mu(f) \in C(G, \mathbb{R}) \subset C(G, \mathbb{C})$  so "locally bounded" in this context means that for each  $a \in G$  there are constants  $M$  and  $r > 0$  such that for all  $f \in \mathcal{F}$  we have  $|\mu(f)(z)| \leq M$  for all  $|z - a| < r$  (technically, "uniformly bounded" is only defined for functions in  $H(G)$  in Section VII.2). To show normality of  $\mathcal{F}$  in  $C(G, \mathbb{C}_\infty)$ , we apply the Arzela-Ascoli Theorem (Theorem VII.1.23). This requires that for each  $z \in G$ ,  $\{f(z) \mid f \in \mathcal{F}\}$  has compact closure in  $\Omega = \mathbb{C}_\infty$ . But  $\mathbb{C}_\infty$  is compact (see the exercises for the supplement "The Extended Complex Plane") and a closed subset of a compact set is compact by Proposition II.4.3(b), so this condition is satisfied. We also need  $\mathcal{F}$  to be equicontinuous at each point of  $G$ , so we now show that.

## Theorem VII.3.4 (continued 4)

**Theorem VII.3.4.** Let  $\{f_n\}$  be a sequence in  $M(G)$  and suppose  $f_n \rightarrow f$  in  $C(G, \mathbb{C}_\infty)$ . Then either  $f$  is meromorphic or  $f \equiv \infty$ . If each  $f_n$  is analytic then either  $f$  is analytic on  $f \equiv \infty$ .

**Proof (continued).** (III) If each  $f_n$  is analytic in  $G$  then each  $f_n$  is finite in  $G$  and so  $1/f_n$  has no zeros in  $G$ . So for any  $a \in G$  and  $B(a; r) \subset G$ ,  $1/f_n$  has no zeros in  $B(a; r)$  and so by Corollary VII.2.6, either  $1/f \equiv 0$  or  $1/f$  never vanishes on  $B(a; r)$ . That is, either  $f \equiv \infty$  or  $f$  is analytic on  $B(a; r)$ . Combining this with Part I we have that either  $f \equiv \infty$  or  $f$  is analytic on  $G$ . □

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## Theorem VII.3.8 (continued 1)

**Theorem VII.3.8.** A family  $\mathcal{F} \subset M(G)$  is normal in  $C(G, \mathbb{C}_\infty)$  if and only if  $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$  is locally bounded.

**Proof (continued).** As in Lemma VII.2.8 (which is in the setting of  $H(G)$ ), pointwise locally bounded implies locally bounded on compact sets. Let  $K$  be an arbitrary closed disk in  $G$  (so  $K$  is closed and bounded and by Heine-Borel Theorem is a compact set). Then the local boundedness of  $\mu(\mathcal{F})$  implies that there is  $M$  such that  $|\mu(f)(z)| = \mu(f)(z) = 2|f'(z)|/(1 + |f(z)|^2) \leq M$  for all  $z \in K$  and for all  $f \in \mathcal{F}$ . Let  $z_1, z' \in K$ .

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## Theorem VII.3.8 (continued 2)

**Proof (continued).** (I) Suppose neither  $z$  nor  $z'$  are poles of a fixed function  $f \in \mathcal{F}$ . Let  $\alpha > 0$ . Choose points  $w_0 = z, w_1, w_2, \dots, w_n = z'$  in  $K$  which satisfy the following:

(3.9) for  $1 \leq k \leq n$ ,  $w \in [w_{k-1}, w_k]$  implies  $w$  is not a pole of  $f$ ;

(3.10)  $\sum_{k=1}^n |w_k - w_{k-1}| < 2|z - z'|$ ;

(3.11)  $\left| \frac{1 + |f(w_{k-1})|^2}{\{(1 + |f(w_k)|^2)(1 + |f(w_{k-1})|^2)\}^{1/2}} - 1 \right| < \alpha$ , for  $1 \leq k \leq n$ ;

(3.12)  $\left| \frac{f(w_k) = f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha$ , for  $a \leq k \leq n$ .

Since  $K$  is a closed disk and the poles of  $f$  are isolated, then there are only finitely many poles of  $f$  in  $K$  (by the Bolzano-Weierstrass Theorem; see

Theorem 2.12 of my Analysis 1, MATH 4217/5217, notes). So there is a polynomial path  $P$  from  $z$  to  $z'$  satisfying (3.9) and (3.10)....

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Theorem VII.3.8

## Theorem VII.3.8 (continued 4)

**Proof (continued).** So for each  $z_1 \in P$ , there is a small disk centered at  $z_1$  such that both

$$|g(z_1, z_2) - 1| < \alpha \text{ and } \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$$

for all  $z_2$  in the small disk. Now the resulting collection of “small disks” (one for each  $z_1 \in P$ ) is an open cover of  $P$ . Since  $P$  is a compact set, there is a finite number of small disks covering  $P$ . Then label points  $w_0 = z, w_1, w_2, \dots, w_n = z'$  (which results in a “refinement” of the original polygon; only straight line segments are refined here) such that  $[w_{k-1}, w_k]$  lies entirely inside a given open disk for  $k = 1, 2, \dots, n$ . Then the polygonal path determined by  $w_0 = z, w_1, w_2, \dots, w_n = z'$  satisfies (3.11) and (3.12) (an properties (3.9) and (3.10) still hold).

## Theorem VII.3.8 (continued 3)

**Proof (continued).** ... (start with a line segment from  $z$  to  $z'$ ; it can contain only finitely many poles, so modify it to go around any poles—this can be done without involving other poles since the poles are isolated).

Now consider

$$g(z_1, z_2) = \frac{1 + |f(z_1)|^2}{\{(1 + |f(z_1)|^2)(1 + |f(z - 1)|^2)\}^{1/2}}.$$

Then  $g$  is continuous on  $K$ , except at the poles of  $f$ . Since  $g(z_1, z_2) = 1$ , then for given  $\alpha$  an  $d\alpha > 0$ , if  $z_2$  is sufficiently close to  $z_1$  then  $|g(z_1, z_2) - 1| < \alpha$ . So for each  $z_1 \in P$  there is a small open disk centered at  $z_1$  such that for all  $z_2$  in the disk,  $|g(z_1, z_2) - 1| < \alpha$ . Since  $f$  is differentiable at each  $z_1 \in P$  then there is a small open disk centered at  $z_1$  such that for all  $z_2$  in the disk,

$$\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha.$$

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Theorem VII.3.8

## Theorem VII.3.8 (continued 5)

**Proof (continued).** With  $\beta_k = \{(1 + |f(w_{k-1})|^2)(1 + |f(w_k)|^2)\}^{1/2}$ , we have

$$d(f(z), f(z')) \leq \sum_{k=1}^n d(f(w_{k-1}), f(w_k)) \text{ by the Triangle Inequality,}$$

and the facts that  $w_0 = z$ , and  $z_n = z'$

$$\begin{aligned} &= \sum_{k=1}^n \frac{2}{\beta_k} |f(w_k) - f(w_{k-1})| \text{ by definition of } d \\ &= \sum_{k=1}^n \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} \right| |w_k - w_{k-1}| \\ &= \sum_{k=1}^n \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) + f'(w_{k-1}) \right| |w_k - w_{k-1}| \end{aligned}$$

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## Theorem VII.3.8 (continued 6)

**Proof (continued).**

$$\begin{aligned}
 &\leq \sum_{k=1}^n \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| |w_k - w_{k-1}| \\
 &\quad + \sum_{k=1}^n \frac{2}{\beta_k} |f'(w_{k-1})| |w_k - w_{k-1}| \\
 &\leq 2\alpha \sum_{k=1}^n k = 1^n \frac{1}{\beta_k} |w_k - w_{k-1}| + \sum_{k=1}^n \frac{2}{\beta_k} |f'(w_{k-1})| |w_k - w_{k-1}| \\
 &\quad \text{by (3.11)} \\
 &\leq 2\alpha \sum_{k=1}^n \frac{1}{\beta_k} |w_k - w_{k-1}| + M \sum_{k=1}^n \frac{1 + |f(w_{k-1})|^2}{\beta_k} |w_k - w_{k-1}| \\
 &\quad \text{since } \mu(f)(w_{k-1}) = \frac{2|f'(w_{k-1})|}{1 + |f(w_{k-1})|^2} \leq M \\
 &\quad \text{by the local boundedness of } \mu(\mathcal{F})
 \end{aligned}$$

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## Theorem VII.3.8 (continued 8)

**Proof (continued).** (II) Now suppose  $z'$  is a pole of  $f$  but  $x$  is not. If  $w \in K$  is not a pole then

$$\begin{aligned}
 d(f(z), \infty) &\leq d(f(z), f(w)) + d(f(w), \infty) \text{ by the Triangle Inequality} \\
 &\leq 2M|z - w| + d(f(w), \infty) \text{ by (3.13)}.
 \end{aligned}$$

Since the poles of  $f$  are isolated, for  $w$  "sufficiently close to"  $z'$ ,  $w$  is not a pole and so  $\lim_{w \rightarrow z'} f(w) = f(z') = \infty$  and  $\lim_{z \rightarrow z'} |z - w| = z - z'$ .

Therefore

$$\begin{aligned}
 d(f(z), f(z')) &= d(f(z), \infty) \leq \lim_{w \rightarrow z'} (2M|z - w| + d(f(w), \infty)) \\
 &= 2M|z - z'| + d(f(z'), \infty) = 2M|z - z'| + d(\infty, \infty) \\
 &= 2M|z - z'| + 0 = 2M|z - z'|.
 \end{aligned}$$

Therefore (3.13) holds if at most one of  $z$  and  $z'$  is a pole.

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## Theorem VII.3.8 (continued 7)

**Proof (continued).**

$$\begin{aligned}
 &\leq 2\alpha \sum_{k=1}^n |w_k - w_{k-1}| + M \sum_{k=1}^n \left( \frac{1 + |f(w_{k-1})|^2}{\beta_k} - 1 + 1 \right) |w_k - w_{k-1}| \\
 &\quad \text{since } \beta_k = \{(1 + |f(w_{k-1})|^2)((1 + |f(w_k)|^2)^{1/2} \geq 1 \text{ and } 1/\beta_k \leq 1 \\
 &\leq 4\alpha|z - z'| + M \sum_{k=1}^n |w_k - w_{k-1}| + M \sum_{k=1}^n \frac{1 + |f(w_{k-1})|^2}{\beta_k} |w_k - w_{k-1}| \\
 &\quad \text{by the Triangle Inequality and (3.10)} \\
 &\leq 4\alpha|z - z'| + 2M|z - z'| + M\alpha(2|z - z'|) \text{ by (3.10) and (3.12)} \\
 &= (4\alpha + 2M\alpha + 2M)|z - z'|.
 \end{aligned}$$

Since  $\alpha > 0$  is arbitrary an  $dz, z'$  are any nonpoles of  $f$ , then

$$d(f(z), f(z')) \leq 2M|z - z'| \text{ for nonpoles } z, z' \in G. \quad (3.13)$$

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## Theorem VII.3.8 (continued 9)

**Proof (continued).** (III) Similarly, if  $z$  and  $z'$  are both poles, then for  $w$  "sufficiently close to";  $z$  is not a pole of  $f$  (since poles are isolated) and so

$$\begin{aligned}
 (f(z), f(z')) &= d\left(\lim_{w \rightarrow z} f(w), f(z')\right) \\
 &= \lim_{w \rightarrow z} (f(w), f(z')) \leq \lim_{w \rightarrow z} 2M|w - z'| \text{ by Part II} \\
 &= 2M|z - z'|
 \end{aligned}$$

and (3.13) holds if both  $z$  and  $z'$  are poles of  $f$  (in fact, this holds trivially since  $f(z) = f(z') = \infty$  and so  $d(f(z), f(z')) = 0$ ). Therefore, (3.13) holds for all  $z, z' \in K$ .

At this stage, we have that given any closed disk  $K \subset G$  that for all  $z, z' \in K$ ,  $d(f(z), f(z')) \leq 2M|z - z'|$  for all  $f \in \mathcal{F}$  (since this conclusion holds for arbitrary  $f \in \mathcal{F}$ , as shown in Parts I, II, and III).

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## Theorem VII.3.8 (continued 10)

**Theorem VII.3.8.** A family  $\mathcal{F} \subset M(G)$  is normal in  $C(G, \mathbb{C}_\infty)$  if and only if  $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$  is locally bounded.

**Proof (continued).** Let  $a \in G$ . Let  $K = B(a; r) \subset G$  and let  $\varepsilon > 0$ . Let  $M > 0$  be the constant such that  $|\mu(f)(z)| = \mu(f)(z) \leq M$  for all  $z \in K$  and for all  $f \in \mathcal{F}$  (given by the local boundedness of  $\mu(f)$  on  $K$ , as argued above before Part I). Define  $\delta = \min\{r, \varepsilon/(2M)\}$ . Then for  $|z - a| < \delta$  we have  $d(f(z), f(a)) < 2M|z - a| < 2M\delta < \varepsilon$  for all  $f \in \mathcal{F}$ . That is,  $\mathcal{F}$  is equicontinuous at point  $a \in G$ . Since  $a \in G$  is an arbitrary point of  $G$ , then by the Arzela-Ascoli Theorem (Theorem VII.4.23),  $\mathcal{F} \subset C(G, \Omega) = C(G, \mathbb{C}_\infty)$  is normal

The converse is to be given in Exercise VII.3.2. □