Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.3. Spaces of Meromorphic Functions—Proofs of Theorems

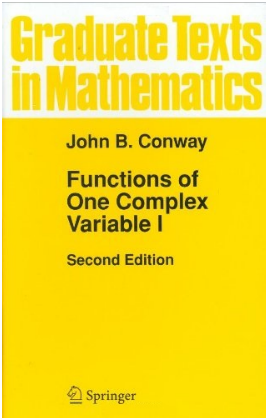

Table of contents

1. Theorem VII.3.4

2. Theorem VII.3.8

Theorem VII.3.4

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof. (I) Let $a \in G$ with $f(a) \neq \infty$. Set $M = |f(a)|$. By Proposition VII.3.3(a), there is $\rho > 0$ such that $B_\infty(f(a); \rho) \subset B(f(z); M)$.

Theorem VII.3.4

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof. (I) Let $a \in G$ with $f(a) \neq \infty$. Set $M = |f(a)|$. By Proposition VII.3.3(a), there is $\rho > 0$ such that $B_\infty(f(a); \rho) \subset B(f(z); M)$. Since $f_n \to f$ then there is $n_0 \in \mathbb{N}$ such that $d(f_n(a), f(a)) < \rho/2$ for all $n \geq n_0$ (since convergence in $C(G, \mathbb{C}_\infty)$ implies convergence on compact subsets of G and so implies pointwise convergence since $\{a\} \subset G \subset \mathbb{C}$ is compact). Now $\{f-, f_1, f_2, \ldots\} \subset C(G, \mathbb{C}_\infty)$ is compact since any open cover must include an open set containing f and this open set contains all by finitely many of the other elements of the set.

Theorem VII.3.4

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof. (I) Let $a \in G$ with $f(a) \neq \infty$. Set $M = |f(a)|$. By Proposition VII.3.3(a), there is $\rho > 0$ such that $B_\infty(f(a); \rho) \subset B(f(z); M)$. Since $f_n \to f$ then there is $n_0 \in \mathbb{N}$ such that $d(f_n(a), f(a)) < \rho/2$ for all $n \geq n_0$ (since convergence in $C(G, \mathbb{C}_\infty)$ implies convergence on compact subsets of G and so implies pointwise convergence since $\{a\} \subset G \subset \mathbb{C}$ is compact). Now $\{f-, f_1, f_2, \ldots\} \subset C(G, \mathbb{C}_\infty)$ is compact since any open cover must include an open set containing f and this open set contains all by finitely many of the other elements of the set. So $\{f, f_1, f_2, \ldots\}$ has compact closure and by Proposition VII.1.15 it is normal. By the Arzela-Ascoli Theorem (Theorem VII.1.23(b)) $\{f, f_1, f_2, \ldots\}$ is equicontinuous. So by the definition of equicontinuity, there is $r_1 > 0$ such that $|z - a| < r_1$ implies $d(f_n(z), f_n(z)) < \rho/2$.

Theorem VII.3.4

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof. (I) Let $a \in G$ with $f(a) \neq \infty$. Set $M = |f(a)|$. By Proposition VII.3.3(a), there is $\rho > 0$ such that $B_\infty(f(a); \rho) \subset B(f(z); M)$. Since $f_n \to f$ then there is $n_0 \in \mathbb{N}$ such that $d(f_n(a), f(a)) < \rho/2$ for all $n \geq n_0$ (since convergence in $C(G, \mathbb{C}_\infty)$ implies convergence on compact subsets of G and so implies pointwise convergence since $\{a\} \subset G \subset \mathbb{C}$ is compact). Now $\{f-, f_1, f_2, \ldots\} \subset C(G, \mathbb{C}_\infty)$ is compact since any open cover must include an open set containing f and this open set contains all by finitely many of the other elements of the set. So $\{f, f_1, f_2, \ldots\}$ has compact closure and by Proposition VII.1.15 it is normal. By the Arzela-Ascoli Theorem (Theorem VII.1.23(b)) $\{f, f_1, f_2, \ldots\}$ is equicontinuous. So by the definition of equicontinuity, there is $r_1 > 0$ such that $|z - a| < r_1$ implies $d(f_n(z), f_n(z)) < \rho/2$.

Theorem VII.3.4 (continued 1)

Proof (continued). So for $|z-a| \leq r < r_1$ and $n \geq n_0$ we have

$$d(f_n(z), f(a)) \leq d(f_n(z), f_n(a)) + d(f_n(a), f(a)) < \rho/2 + \rho/2 = \rho.$$

By the choice of ρ, $B_\infty(f(a);\rho) \subset B(f(a); M)$, so $d(f_n(z), f(a)) < \rho$ implies $f_n(z) \in B_\infty(f(a);\rho) \subset B(f(a); M)$ and so $|f_n(z) - f(a)| < M$. So for $z \in \overline{B}(a; r)$ and $n \geq n_0$ we have

$$|f_n(z)| = |f_n(z) - f(a) + f(a)| \leq |f_n(z) - f(a)| + |f(a)| < M + M = 2M.$$

Since $f_n(z) \to f(z)$, then $f(z) \leq 2M$ for all $z \in \overline{B}(a; r)$. So

$$d(f_n(z), f(z)) = \frac{2|f_n(z) - f(z)|}{\{(1+|f_n(z)|^2)(1+|f(z)|^2)\}^{1/2}}$$

$$\geq \frac{2|f_n(z) - f(z)|}{\{(1+(2M)^2)(1+(2M)^2)\}^{1/2}} = \frac{2|f_n(z) - f(z)|}{1+4M^2}$$

for $z \in \overline{B}(a; r)$ and $n \geq n_0$.

Theorem VII.3.4 (continued 1)

Proof (continued). So for $|z-a| \leq r < r_1$ and $n \geq n_0$ we have

$$d(f_n(z), f(a)) \leq d(f_n(z), f_n(a)) + d(f_n(a), f(a)) < \rho/2 + \rho/2 = \rho.$$

By the choice of ρ, $B_\infty(f(a);\rho) \subset B(f(a);M)$, so $d(f_n(z), f(a)) < \rho$ implies $f_n(z) \in B_\infty(f(a);\rho) \subset B(f(a);M)$ and so $|f_n(z) - f(a)| < M$. So for $z \in \overline{B}(a;r)$ and $n \geq n_0$ we have

$$|f_n(z)| = |f_n(z) - f(a) + f(a)| \leq |f_n(z) - f(a)| + |f(a)| < M + M = 2M.$$

Since $f_n(z) \to f(z)$, then $f(z) \leq 2M$ for all $z \in \overline{B}(a;r)$. So

$$d(f_n(z), f(z)) = \frac{2|f_n(z) - f(z)|}{\{(1+|f_n(z)|^2)(1+|f(z)|^2)\}^{1/2}}$$

$$\geq \frac{2|f_n(z) - f(z)|}{\{(1+(2M)^2)(1+(2M)^2)\}^{1/2}} = \frac{2|f_n(z) - f(z)|}{1+4M^2}$$

for $z \in \overline{B}(a;r)$ and $n \geq n_0$.

Theorem VII.3.4 (continued 2)

.

Proof (continued). By Proposition VII.1.10, $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies that $d(f_n(z), f(z)) \to 0$ uniformly on cmpact set $\overline{B}(a; r)$. So the above inequality implies that $|f_n(z) - f(z)| \to 0$ uniformly in $\overline{B}(a; r)$. Now for $n \geq n_0$, $|f_n(z)| \leq 2M$ for all $z \in \overline{B}(a; r)$ so for $n \geq n_0$ each meromorphic f_n must be analytic on $B(a; r)$ (since bounded f_n cannot have a pole in $B(a; r)$). So $\{f_n\}_{n \geq n_0}$ is a sequence of analytic functions which converges uniformly on $\overline{B}(a; r)$ (and so converges uniformly on any compact subset of $\overline{B}(a; r)$) and then by Proposition VII.1.10 $\{f_n\}_{n \geq n_0}$ converges in $C(B(a; r), \mathbb{C}_\infty)$. By Theorem VII.2.1, the limit function f is analytic on $B(a; r)$. That is, if $f(a) \neq \infty$ then f is analytic in some neighborhood of a.

Theorem VII.3.4 (continued 2)

.

Proof (continued). By Proposition VII.1.10, $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies that $d(f_n(z), f(z)) \to 0$ uniformly on cmpact set $\overline{B}(a; r)$. So the above inequality implies that $|f_n(z) - f(z)| \to 0$ uniformly in $\overline{B}(a; r)$. Now for $n \geq n_0$, $|f_n(z)| \leq 2M$ for all $z \in \overline{B}(a; r)$ so for $n \geq n_0$ each meromorphic f_n must be analytic on $B(a; r)$ (since bounded f_n cannot have a pole in $B(a; r)$). So $\{f_n\}_{n \geq n_0}$ is a sequence of analytic functions which converges uniformly on $\overline{B}(a; r)$ (and so converges uniformly on any compact subset of $\overline{B}(a; r)$) and then by Proposition VII.1.10 $\{f_n\}_{n \geq n_0}$ converges in $C(B(a; r), \mathbb{C}_\infty)$. By Theorem VII.2.1, the limit function f is analytic on $B(a; r)$. That is, if $f(a) \neq \infty$ then f is analytic in some neighborhood of a.

(II) Now suppose there is $a \in G$ with $f(a) = \infty$. For $g \in C(G, \mathbb{C}_\infty)$, define $\left(\frac{1}{g}\right) = \begin{cases} 1/g(z) & \text{if } g(z) \notin \{0, \infty\} \\ 0 & \text{if } g(z) = \infty \\ \infty & \text{if } g(z) = 0. \end{cases}$ Then $1/g \in C(G, \mathbb{C}_\infty)$.

Theorem VII.3.4 (continued 2)

.

Proof (continued). By Proposition VII.1.10, $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies that $d(f_n(z), f(z)) \to 0$ uniformly on cmpact set $\overline{B}(a; r)$. So the above inequality implies that $|f_n(z) - f(z)| \to 0$ uniformly in $\overline{B}(a; r)$. Now for $n \geq n_0$, $|f_n(z)| \leq 2M$ for all $z \in \overline{B}(a; r)$ so for $n \geq n_0$ each meromorphic f_n must be analytic on $B(a; r)$ (since bounded f_n cannot have a pole in $B(a; r)$). So $\{f_n\}_{n \geq n_0}$ is a sequence of analytic functions which converges uniformly on $\overline{B}(a; r)$ (and so converges uniformly on any compact subset of $\overline{B}(a; r)$) and then by Proposition VII.1.10 $\{f_n\}_{n \geq n_0}$ converges in $C(B(a; r), \mathbb{C}_\infty)$. By Theorem VII.2.1, the limit function f is analytic on $B(a; r)$. That is, if $f(a) \neq \infty$ then f is analytic in some neighborhood of a.

(II) Now suppose there is $a \in G$ with $f(a) = \infty$. For $g \in C(G, \mathbb{C}_\infty)$, define $\left(\frac{1}{g}\right) = \begin{cases} 1/g(z) & \text{if } g(z) \notin \{0, \infty\} \\ 0 & \text{if } g(z) = \infty \\ \infty & \text{if } g(z) = 0. \end{cases}$ Then $1/g \in C(G, \mathbb{C}_\infty)$.

Theorem VII.3.4 (continued 3)

Proof (continued). Since, as observed above, $d(z_1, z_2) = d(1/z_1, 1/z_2)$ and $d(z, 0) = d(1/z, \infty)$ for $z \neq 0$, then $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies $1/f_n \to 1/f$ in $C(G, \mathbb{C}_\infty)$. Since each f_n is meromorphic on G, then each $1/f_n$ is meromorphic on G. Since $(1/f)(z) = 0 \neq \infty$, by Part I there is $r > 0$ and $n_0 \in \mathbb{N}$ such that $1/f$ and $1/f_n$ are analytic on $B(a; r)$ for $n \geq n_0$, and $1/f_n \to 1/f$ uniformly on $B(a; r)$. So by Proposition VII.1.10(b), $1/f_n \to 1/f$ in $H(G) \subset C(G, \mathbb{C})$.

Theorem VII.3.4 (continued 3)

Proof (continued). Since, as observed above, $d(z_1, z_2) = d(1/z_1, 1/z_2)$ and $d(z, 0) = d(1/z, \infty)$ for $z \neq 0$, then $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies $1/f_n \to 1/f$ in $C(G, \mathbb{C}_\infty)$. Since each f_n is meromorphic on G, then each $1/f_n$ is meromorphic on G. Since $(1/f)(z) = 0 \neq \infty$, by Part I there is $r > 0$ and $n_0 \in \mathbb{N}$ such that $1/f$ and $1/f_n$ are analytic on $B(a; r)$ for $n \geq n_0$, and $1/f_n \to 1/f$ uniformly on $B(a; r)$. So by Proposition VII.1.10(b), $1/f_n \to 1/f$ in $H(G) \subset C(G, \mathbb{C})$. So the hypotheses of Hurwitz's Theorem (Theorem VII.2.5) and the corollary to Hurwitz's Theorem, Corollary VII.2.6, are satisfied. Since $1/f$ is analytic on $B(a; r)$ and $(1/f)(a) = 0$, then by Corollary IV.3.10 then either $1/f \equiv 0$ or there is $R > 0$ such that $B(a; R) \subset G$ and $(1/f)(z) \neq 0$ for $0 < |z - a| < R$; that is, either $a/f \equiv 0$ or $z = a$ is an isolated zero of $1/f$. Therefore, either $f \equiv 0$ or f is meromorphic on $B(a; R)$. (Conway uses Hurwitz's Theorem to reach this conclusion.)

Theorem VII.3.4 (continued 3)

Proof (continued). Since, as observed above, $d(z_1, z_2) = d(1/z_1, 1/z_2)$ and $d(z, 0) = d(1/z, \infty)$ for $z \neq 0$, then $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies $1/f_n \to 1/f$ in $C(G, \mathbb{C}_\infty)$. Since each f_n is meromorphic on G, then each $1/f_n$ is meromorphic on G. Since $(1/f)(z) = 0 \neq \infty$, by Part I there is $r > 0$ and $n_0 \in \mathbb{N}$ such that $1/f$ and $1/f_n$ are analytic on $B(a; r)$ for $n \geq n_0$, and $1/f_n \to 1/f$ uniformly on $B(a; r)$. So by Proposition VII.1.10(b), $1/f_n \to 1/f$ in $H(G) \subset C(G, \mathbb{C})$. So the hypotheses of Hurwitz's Theorem (Theorem VII.2.5) and the corollary to Hurwitz's Theorem, Corollary VII.2.6, are satisfied. Since $1/f$ is analytic on $B(a; r)$ and $(1/f)(a) = 0$, then by Corollary IV.3.10 then either $1/f \equiv 0$ or there is $R > 0$ such that $B(a; R) \subset G$ and $(1/f)(z) \neq 0$ for $0 < |z - a| < R$; that is, either $a/f \equiv 0$ or $z = a$ is an isolated zero of $1/f$. Therefore, either $f \equiv 0$ or f is meromorphic on $B(a; R)$. (Conway uses Hurwitz's Theorem to reach this conclusion.)

Combining Parts I and II, either $f \equiv \infty$ or f is meromorphic in G; that is, $f \in M(G) \cup \{\infty\}$.

Theorem VII.3.4 (continued 3)

Proof (continued). Since, as observed above, $d(z_1, z_2) = d(1/z_1, 1/z_2)$ and $d(z, 0) = d(1/z, \infty)$ for $z \neq 0$, then $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ implies $1/f_n \to 1/f$ in $C(G, \mathbb{C}_\infty)$. Since each f_n is meromorphic on G, then each $1/f_n$ is meromorphic on G. Since $(1/f)(z) = 0 \neq \infty$, by Part I there is $r > 0$ and $n_0 \in \mathbb{N}$ such that $1/f$ and $1/f_n$ are analytic on $B(a; r)$ for $n \geq n_0$, and $1/f_n \to 1/f$ uniformly on $B(a; r)$. So by Proposition VII.1.10(b), $1/f_n \to 1/f$ in $H(G) \subset C(G, \mathbb{C})$. So the hypotheses of Hurwitz's Theorem (Theorem VII.2.5) and the corollary to Hurwitz's Theorem, Corollary VII.2.6, are satisfied. Since $1/f$ is analytic on $B(a; r)$ and $(1/f)(a) = 0$, then by Corollary IV.3.10 then either $1/f \equiv 0$ or there is $R > 0$ such that $B(a; R) \subset G$ and $(1/f)(z) \neq 0$ for $0 < |z - a| < R$; that is, either $a/f \equiv 0$ or $z = a$ is an isolated zero of $1/f$. Therefore, either $f \equiv 0$ or f is meromorphic on $B(a; R)$. (Conway uses Hurwitz's Theorem to reach this conclusion.)

Combining Parts I and II, either $f \equiv \infty$ or f is meromorphic in G; that is, $f \in M(G) \cup \{\infty\}$.

Theorem VII.3.4 (continued 4)

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof (continued). (III) If each f_n is analytic in G then each f_n is finite in G and so $1/f_n$ has no zeros in G. So for any $a \in G$ and $B(a;r) \subset G$, $1/f_n$ has no zeros in $B(a;r)$ and so by Corollary VII.2.6, either $1/f \equiv 0$ or $1/f$ never vanishes on $B(a;r)$. That is, either $f \equiv \infty$ or f is analytic on $B(a;r)$. Combining this with Part I we have that either $f \equiv \infty$ or f is analytic on G. □

Theorem VII.3.4 (continued 4)

Theorem VII.3.4. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic on $f \equiv \infty$.

Proof (continued). (III) If each f_n is analytic in G then each f_n is finite in G and so $1/f_n$ has no zeros in G. So for any $a \in G$ and $B(a;r) \subset G$, $1/f_n$ has no zeros in $B(a;r)$ and so by Corollary VII.2.6, either $1/f \equiv 0$ or $1/f$ never vanishes on $B(a;r)$. That is, either $f \equiv \infty$ or f is analytic on $B(a;r)$. Combining this with Part I we have that either $f \equiv \infty$ or f is analytic on G. □

Theorem VII.3.8

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof of locally bounded implies normal. Let $\mathcal{F} \subset M(G)$ with $\mu(\mathcal{F})$ locally bounded. Notice that $\mu(f) \in C(G, \mathbb{R}) \subset C(G, \mathbb{C})$ so "locally bounded" in this context means that for each $a \in G$ there are constants M and $r > 0$ such that for all $f \in \mathcal{F}$ we have $|\mu(f)(z)| \leq M$ for all $|z - a| < r$ (technically, "uniformly bounded" is only defined for functions in $H(G)$ in Section VII.2).

Theorem VII.3.8

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof of locally bounded implies normal. Let $\mathcal{F} \subset M(G)$ with $\mu(\mathcal{F})$ locally bounded. Notice that $\mu(f) \in C(G, \mathbb{R}) \subset C(G, \mathbb{C})$ so "locally bounded" in this context means that for each $a \in G$ there are constants M and $r > 0$ such that for all $f \in \mathcal{F}$ we have $|\mu(f)(z)| \leq M$ for all $|z - a| < r$ (technically, "uniformly bounded" is only defined for functions in $H(G)$ in Section VII.2). To show normality of $\mathcal{F}$ in $C(G, \mathbb{C}_\infty)$, we apply the Arzela-Ascoli Theorem (Theorem VII.1.23). This requires that for each $z \in G$, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in $\Omega = \mathbb{C}_\infty$. But $\mathbb{C}_\infty$ is compact (see the exercises for the supplement "The Extended Complex Plane") and a closed subset of a compact set is compact by Proposition II.4.3(b), so this condition is satisfied. We also need $\mathcal{F}$ to be equicontinuous at each point of G, so we now show that.

Theorem VII.3.8

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof of locally bounded implies normal. Let $\mathcal{F} \subset M(G)$ with $\mu(\mathcal{F})$ locally bounded. Notice that $\mu(f) \in C(G, \mathbb{R}) \subset C(G, \mathbb{C})$ so "locally bounded" in this context means that for each $a \in G$ there are constants M and $r > 0$ such that for all $f \in \mathcal{F}$ we have $|\mu(f)(z)| \le M$ for all $|z - a| < r$ (technically, "uniformly bounded" is only defined for functions in $H(G)$ in Section VII.2). To show normality of $\mathcal{F}$ in $C(G, \mathbb{C}_\infty)$, we apply the Arzela-Ascoli Theorem (Theorem VII.1.23). This requires that for each $z \in G$, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in $\Omega = \mathbb{C}_\infty$. But $\mathbb{C}_\infty$ is compact (see the exercises for the supplement "The Extended Complex Plane") and a closed subset of a compact set is compact by Proposition II.4.3(b), so this condition is satisfied. We also need $\mathcal{F}$ to be equicontinuous at each point of G, so we now show that.

Theorem VII.3.8 (continued 1)

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof (continued). As in Lemma VII.2.8 (which is in the setting of $H(G)$), pointwise locally bounded implies locally bounded on compact sets. Let K be an arbitrary closed disk in G (so K is closed and bounded and by Heine-Borel Theorem is a compact set). Then the local boundedness of $\mu(\mathcal{F})$ implies that there is M such that $|\mu(f)(z)| = \mu(f)(z) = 2|f'(z)|/(1 + |f(z)|^2) \leq M$ for all $z \in K$ and for all $f \in \mathcal{F}$. Let $z_1, z' \in K$.

Theorem VII.3.8 (continued 1)

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof (continued). As in Lemma VII.2.8 (which is in the setting of $H(G)$), pointwise locally bounded implies locally bounded on compact sets. Let K be an arbitrary closed disk in G (so K is closed and bounded and by Heine-Borel Theorem is a compact set). Then the local boundedness of $\mu(\mathcal{F})$ implies that there is M such that $|\mu(f)(z)| = \mu(f)(z) = 2|f'(z)|/(1 + |f(z)|^2) \leq M$ for all $z \in K$ and for all $f \in \mathcal{F}$. Let $z_1, z' \in K$.

Theorem VII.3.8 (continued 2)

Proof (continued). (I) Suppose neither z nor z' are poles of a fixed function $f \in \mathcal{F}$. Let $\alpha > 0$. Choose points $w_0 = z, w_1, w_2, \ldots, w_n = z'$ in K which satisfy the following:

(3.9) for $1 \leq k \leq n$, $w \in [w_{k-1}, w_k]$ implies w is not a pole of f;

(3.10) $\sum_{k=1}^n |w_k - w_{k-1}| < 2|z - z'|$;

(3.11) $\left| \dfrac{1 + |f(w_{k-1})|^2}{\{(1 + |f(w_k)|^2)(1 + |f(w_{k-1})|^2)\|^{1/2}} - 1 \right| < \alpha$, for $1 \leq k \leq n$;

(3.12) $\left| \dfrac{f(w_k) = f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha$, for $a \leq k \leq n$.

Theorem VII.3.8 (continued 2)

Proof (continued). (I) Suppose neither z nor z' are poles of a fixed function $f \in \mathcal{F}$. Let $\alpha > 0$. Choose points $w_0 = z, w_1, w_2, \ldots, w_n = z'$ in K which satisfy the following:

(3.9) for $1 \leq k \leq n$, $w \in [w_{k-1}, w_k]$ implies w is not a pole of f;

(3.10) $\sum_{k=1}^n |w_k - w_{k-1}| < 2|z - z'|$;

(3.11) $\left| \dfrac{1 + |f(w_{k-1})|^2}{\{(1 + |f(w_k)|^2)(1 + |f(w_{k-1})|^2)\|^{1/2}} - 1 \right| < \alpha$, for $1 \leq k \leq n$;

(3.12) $\left| \dfrac{f(w_k) = f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha$, for $a \leq k \leq n$.

Since K is a closed disk and the poles of f are isolated, then there are only finitely many poles of f in K (by the Bolzano-Weierstrass Theorem; see Theorem 2.12 of my Analysis 1, MATH 4217/5217, notes). So there is a polynomial path P from z to z' satisfying (3.9) and (3.10)...

Theorem VII.3.8 (continued 2)

Proof (continued). (I) Suppose neither z nor z' are poles of a fixed function $f \in \mathcal{F}$. Let $\alpha > 0$. Choose points $w_0 = z, w_1, w_2, \ldots, w_n = z'$ in K which satisfy the following:

(3.9) for $1 \le k \le n$, $w \in [w_{k-1}, w_k]$ implies w is not a pole of f;

(3.10) $\sum_{k=1}^n |w_k - w_{k-1}| < 2|z - z'|$;

(3.11) $\left| \dfrac{1 + |f(w_{k-1})|^2}{\{(1 + |f(w_k)|^2)(1 + |f(w_{k-1})|^2)\|^{1/2}} - 1 \right| < \alpha$, for $1 \le k \le n$;

(3.12) $\left| \dfrac{f(w_k) = f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha$, for $a \le k \le n$.

Since K is a closed disk and the poles of f are isolated, then there are only finitely many poles of f in K (by the Bolzano-Weierstrass Theorem; see Theorem 2.12 of my Analysis 1, MATH 4217/5217, notes). So there is a polynomial path P from z to z' satisfying (3.9) and (3.10)...

Theorem VII.3.8 (continued 3)

Proof (continued). . . . (start with a line segment from z to z'; it can contain only finitely many poles, so modify it to go around any poles—this can be done without involving other poles since the poles are isolated). Now consider

$$g(z_1, z_2) = \frac{1 + |f(z_1)|^2}{\{(1 + |f(z_1)|^2)(1 + |f(z-1)|^2\}^{1/2}}.$$

Then g is continuous on K, except at the poles of f. Since $g(z_1, z_2) = 1$, ten for given a_1 an d$\alpha > 0$, if z_2 is sufficiently close to z_1 then $|g(z_1, z_2) - 1| < \alpha$. So for each $z_1 \in P$ there is a small open disk centered at z_1 such that for all z_2 in the disk, $|g(z_1, z_2) - 1| < \alpha$. Since f is differentiable at each $z_1 \in P$ then there is a small open disk centered at z_1 such that for all z_2 in the disk, $\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$.

Theorem VII.3.8 (continued 3)

Proof (continued). ...(start with a line segment from z to z'; it can contain only finitely many poles, so modify it to go around any poles—this can be done without involving other poles since the poles are isolated). Now consider

$$g(z_1, z_2) = \frac{1 + |f(z_1)|^2}{\{(1 + |f(z_1)|^2)(1 + |f(z-1)|^2\}^{1/2}}.$$

Then g is continuous on K, except at the poles of f. Since $g(z_1, z_2) = 1$, ten for given a_1 an d$\alpha > 0$, if z_2 is sufficiently close to z_1 then $|g(z_1, z_2) - 1| < \alpha$. So for each $z_1 \in P$ there is a small open disk centered at z_1 such that for all z_2 in the disk, $|g(z_1, z_2) - 1| < \alpha$. Since f is differentiable at each $z_1 \in P$ then there is a small open disk centered at z_1 such that for all z_2 in the disk, $\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$.

Theorem VII.3.8 (continued 4)

Proof (continued). So for each $z_1 \in P$, there is a small disk centered at z_1 such that both

$$|g(z_1, z_2) - 1| < \alpha \text{ and } \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$$

for all z_2 in the small disk. Now the resulting collection of "small disks" (one for each $z_1 \in P$) is an open cover of P. Since P is a compact set, there is a finite number of small disks covering P.

Theorem VII.3.8 (continued 4)

Proof (continued). So for each $z_1 \in P$, there is a small disk centered at z_1 such that both

$$|g(z_1, z_2) - 1| < \alpha \text{ and } \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$$

for all z_2 in the small disk. Now the resulting collection of "small disks" (one for each $z_1 \in P$) is an open cover of P. Since P is a compact set, there is a finite number of small disks covering P. Then label points $w_0 = z, w_1, w_2, \ldots, w_n = z'$ (which results in a "refinement" of the original polygon; only straight line segments are refined here) such that $[w_{k-1}, w_k]$ lies entirely inside a given open disk for $k = 1, 2, \ldots, n$. Then the polygonal path determined by $w_0 = z, w_1, w_2, \ldots, w_n = z'$ satisfies (3.11) and (3.12) (an properties (3.9) and (3.10) still hold).

Theorem VII.3.8 (continued 4)

Proof (continued). So for each $z_1 \in P$, there is a small disk centered at z_1 such that both

$$|g(z_1, z_2) - 1| < \alpha \text{ and } \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_1) \right| < \alpha$$

for all z_2 in the small disk. Now the resulting collection of "small disks" (one for each $z_1 \in P$) is an open cover of P. Since P is a compact set, there is a finite number of small disks covering P. Then label points $w_0 = z, w_1, w_2, \ldots, w_n = z'$ (which results in a "refinement" of the original polygon; only straight line segments are refined here) such that $[w_{k-1}, w_k]$ lies entirely inside a given open disk for $k = 1, 2, \ldots, n$. Then the polygonal path determined by $w_0 = z, w_1, w_2, \ldots, w_n = z'$ satisfies (3.11) and (3.12) (an properties (3.9) and (3.10) still hold).

Theorem VII.3.8 (continued 5)

Proof (continued). With $\beta_k = \{(1+|f(w_{k-1})|^2)(1+|f(w_k)|^2)\}^{1/2}$, we have

$$d(f(z), f(z')) \leq \sum_{k=1}^{n} d(f(w_{k-1}), f(w_k)) \text{ by the Triangle Inequality,}$$

$$\text{and the facts that } w_0 = z, \text{ and } z_n = z'$$

$$= \sum_{k=1}^{n} \frac{2}{\beta_k} |f(w_k) - f(w_{k-1})| \text{ by definition of } d$$

$$= \sum_{k=1}^{n} \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} \right| |w_k - w_{k-1}|$$

$$= \sum_{k=1}^{n} \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) + f'(w_{k-1}) \right| |w_k - w_{k-1}|$$

Theorem VII.3.8 (continued 6)

Proof (continued).

$$\leq \sum_{k=1}^{n} \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| |w_k - w_{k-1}|$$

$$+ \sum_{k=1}^{n} \frac{2}{\beta_k} |f'(w_{k-1})||w_k - w_{k-1}|$$

$$\leq 2\alpha \sum k = 1^n \frac{1}{\beta_k} |w_k - w_{k-1}| + \sum_{k=1}^{n} \frac{2}{\beta_k} |f'(w_{k-1})||w_k - w_{k-1}|$$

by (3.11)

$$\leq 2\alpha \sum_{k=1}^{n} \frac{1}{\beta_k} |w_k - w_{k-1}| + M \sum_{k=1}^{n} \frac{1 + |f(w_{k-1})|^2}{\beta_k} |w_k - w_{k-1}|$$

since $\mu(f)(w_{k-1}) = \dfrac{2|f'(w_{k-1})|}{1 + |f(w_{k-1})|^2} \leq M$

by the local boundedness of $\mu(\mathcal{F})$

Theorem VII.3.8 (continued 7)

Proof (continued).

$$\leq \quad 2\alpha \sum_{k=1}^{n} |w_k - w_{k-1}| + M \sum_{k=1}^{n} \left(\frac{1 + |f(w_{k-1})|^2}{\beta_k} - 1 + 1 \right) |w_k - w_{k-1}|$$

since $\beta_k = \{(1 + |f(w_{k-1})|^2)((1 + |f(w_k)|^2)\}^{1/2} \geq 1$ and $1/\beta_k \leq 1$

$$\leq \quad 4\alpha|z - z'| + M \sum_{k=1}^{n} |w_k - w_{k-1} + M \sum_{k=1}^{n} \frac{1 + |f(w_{k-1})|^2}{\beta_k} |w_k - w_{k-1}|$$

by the Triangle Inequality and (3.10)

$$\leq \quad 4\alpha|z - z'| + 2M|z - z'| + M\alpha(2|z - z'|) \text{ by (3.10) and (3.12)}$$

$$= \quad (4\alpha + 2M\alpha + 2M)|z - z'|.$$

Since $\alpha > 0$ is arbitrary an dz, z' are any nonpoles of f, then

$$d(f(z), f(z')) \leq 2M|z - z'| \text{ for nonpoles } z, z' \in G. \qquad (3.13)$$

Theorem VII.3.8 (continued 7)

Proof (continued).

$$\begin{aligned}
&\leq 2\alpha\sum_{k=1}^{n}|w_k - w_{k-1}| + M\sum_{k=1}^{n}\left(\frac{1+|f(w_{k-1})|^2}{\beta_k} - 1 + 1\right)|w_k - w_{k-1}| \\
&\quad \text{since } \beta_k = \{(1+|f(w_{k-1})|^2)((1+|f(w_k)|^2)\}^{1/2} \geq 1 \text{ and } 1/\beta_k \leq 1 \\
&\leq 4\alpha|z-z'| + M\sum_{k=1}^{n}|w_k - w_{k-1} + M\sum_{k=1}^{n}\frac{1+|f(w_{k-1})|^2}{\beta_k}|w_k - w_{k-1}| \\
&\quad \text{by the Triangle Inequality and (3.10)} \\
&\leq 4\alpha|z-z'| + 2M|z-z'| + M\alpha(2|z-z'|) \text{ by (3.10) and (3.12)} \\
&= (4\alpha + 2M\alpha + 2M)|z-z'|.
\end{aligned}$$

Since $\alpha > 0$ is arbitrary an dz, z' are any nonpoles of f, then

$$d(f(z), f(z')) \leq 2M|z-z'| \text{ for nonpoles } z, z' \in G. \quad (3.13)$$

Theorem VII.3.8 (continued 8)

Proof (continued). (II) Now suppose z' is a pole of f but x is not. If $w \in K$ is not a pole then

$$\begin{aligned} d(f(z),\infty) &\le d(f(z),f(w)) + d(f(w),\infty) \text{ by the Triangle Inequality} \\ &\le 2M|z-w| + d(f(w),\infty) \text{ by (3.13).} \end{aligned}$$

Since the poles of f are isolated, for w "sufficiently close to" z', w is not a pole and so $\lim_{w\to z'} f(w) = f(z') = \infty$ and $\lim_{z\to z'} |z-w| = z - z'|$. Therefore

$$d(f(z),f(z')) = d(f(z),\infty) \le \lim_{w\to z'} (2M|z-w| + d(f(w),\infty))$$

$$= 2M|z-z'| + d(f(z'),\infty = 2M|z-z'| + d(\infty,\infty)$$

$$= 2M|z-z'| + 0 = 2M|z-z'|.$$

Therefore (3.13) holds if at most one of z and z' is a pole.

Theorem VII.3.8 (continued 8)

Proof (continued). (II) Now suppose z' is a pole of f but x is not. If $w \in K$ is not a pole then

$$\begin{aligned} d(f(z),\infty) &\leq d(f(z),f(w)) + d(f(w),\infty) \text{ by the Triangle Inequality} \\ &\leq 2M|z-w| + d(f(w),\infty) \text{ by (3.13).} \end{aligned}$$

Since the poles of f are isolated, for w "sufficiently close to" z', w is not a pole and so $\lim_{w\to z'} f(w) = f(z') = \infty$ and $\lim_{z\to z'} |z-w| = z - z'|$. Therefore

$$d(f(z),f(z')) = d(f(z),\infty) \leq \lim_{w\to z'} (2M|z-w| + d(f(w),\infty))$$

$$= 2M|z-z'| + d(f(z'),\infty = 2M|z-z'| + d(\infty,\infty)$$

$$= 2M|z-z'| + 0 = 2M|z-z'|.$$

Therefore (3.13) holds if at most one of z and z' is a pole.

Theorem VII.3.8 (continued 9)

Proof (continued). (III) Similarly, if z and z' are both poles, then for w "sufficiently close to;; z is not a pole of f (since poles are isolated) and so

$$\begin{aligned}(f(z), f(z')) &= d\left(\lim_{w\to z} f(w), f(z')\right) \\ &= \lim_{w\to z}(f(w), f(z')) \le \lim_{w\to z} 2M|w - z'| \text{ by Part II} \\ &= 2M|z - z'|\end{aligned}$$

and (3.13) holds if both z and z' are poles of f (in fact, this holds trivially since $f(z) = f(z') = \infty$ and so $d(f(z), f(z')) = 0$). Therefore, (3.13) holds for all $z, z' \in K$.

At this stage, we have that given any closed disk $K \subset G$ that for all $z, z' \in K$, $d(f(z), f(z')) \le 2M|z - z'|$ for all $f \in \mathcal{F}$ (since this conclusion holds for arbitrary $f \in \mathcal{F}$, as shown in Parts I, II, and III).

Theorem VII.3.8 (continued 9)

Proof (continued). (III) Similarly, if z and z' are both poles, then for w "sufficiently close to;; z is not a pole of f (since poles are isolated) and so

$$
\begin{aligned}
(f(z), f(z')) &= d\left(\lim_{w \to z} f(w), f(z')\right) \\
&= \lim_{w \to z} (f(w), f(z')) \leq \lim_{w \to z} 2M|w - z'| \text{ by Part II} \\
&= 2M|z - z'|
\end{aligned}
$$

and (3.13) holds if both z and z' are poles of f (in fact, this holds trivially since $f(z) = f(z') = \infty$ and so $d(f(z), f(z')) = 0$). Therefore, (3.13) holds for all $z, z' \in K$.

At this stage, we have that given any closed disk $K \subset G$ that for all $z, z' \in K$, $d(f(z), f(z')) \leq 2M|z - z'|$ for all $f \in \mathcal{F}$ (since this conclusion holds for arbitrary $f \in \mathcal{F}$, as shown in Parts I, II, and III).

Theorem VII.3.8 (continued 10)

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof (continued). Let $a \in G$. Let $K = B(a; r) \subset G$ and let $\varepsilon > 0$. Let $M > 0$ be the constant such that $|\mu(f)(z)| = \mu(f)(z) \le M$ for all $z \in K$ and for all $f \in \mathcal{F}$ (given by the local boundedness of $\mu(f)$ on K, as argued above before Part I). Define $\delta = \min\{r, \varepsilon/(2M)\}$. Then for $|z - a| < \delta$ we have $d(f(z), f(a)) < 2M|z - a| < 2M\delta < \varepsilon$ for all $f \in \mathcal{F}$. That is, $\mathcal{F}$ is equicontinuous at point $a \in G$. Since $a \in G$ is an arbitrary point of G, then by the Arzela-Ascoli Theorem (Theorem VII.4.23), $\mathcal{F} \subset C(G, \Omega) = C(G, \mathbb{C}_\infty)$ is normal

Theorem VII.3.8 (continued 10)

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof (continued). Let $a \in G$. Let $K = B(a; r) \subset G$ and let $\varepsilon > 0$. Let $M > 0$ be the constant such that $|\mu(f)(z)| = \mu(f)(z) \leq M$ for all $z \in K$ and for all $f \in \mathcal{F}$ (given by the local boundedness of $\mu(f)$ on K, as argued above before Part I). Define $\delta = \min\{r, \varepsilon/(2M)\}$. Then for $|z - a| < \delta$ we have $d(f(z), f(a)) < 2M|z - a| < 2M\delta < \varepsilon$ for all $f \in \mathcal{F}$. That is, $\mathcal{F}$ is equicontinuous at point $a \in G$. Since $a \in G$ is an arbitrary point of G, then by the Arzela-Ascoli Theorem (Theorem VII.4.23), $\mathcal{F} \subset C(G, \Omega) = C(G, \mathbb{C}_\infty)$ is normal

The converse is to be given in Exercise VII.3.2. □

Theorem VII.3.8 (continued 10)

Theorem VII.3.8. A family $\mathcal{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathcal{F}) = \{\mu(f) \mid f \in \mathcal{F}\}$ is locally bounded.

Proof (continued). Let $a \in G$. Let $K = B(a; r) \subset G$ and let $\varepsilon > 0$. Let $M > 0$ be the constant such that $|\mu(f)(z)| = \mu(f)(z) \leq M$ for all $z \in K$ and for all $f \in \mathcal{F}$ (given by the local boundedness of $\mu(f)$ on K, as argued above before Part I). Define $\delta = \min\{r, \varepsilon/(2M)\}$. Then for $|z - a| < \delta$ we have $d(f(z), f(a)) < 2M|z - a| < 2M\delta < \varepsilon$ for all $f \in \mathcal{F}$. That is, $\mathcal{F}$ is equicontinuous at point $a \in G$. Since $a \in G$ is an arbitrary point of G, then by the Arzela-Ascoli Theorem (Theorem VII.4.23), $\mathcal{F} \subset C(G, \Omega) = C(G, \mathbb{C}_\infty)$ is normal

The converse is to be given in Exercise VII.3.2. □