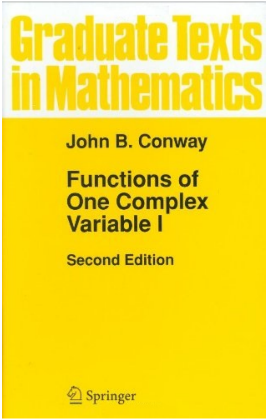
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.4. The Riemann Mapping Theorem—Proofs of Theorems



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$$\begin{cases} \Gamma(s,0) = \gamma_1(s) \text{ and } \Gamma(s,1) = c \text{ for } s \in [0,1] \\ \Gamma(0,t) = \Gamma(1,t) \text{ for } t \in [0,1] \end{cases}$$

for some constant $c \in G_1$.

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Lemma VII.4.A (continued)

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Proof (continued). But then $f \circ \Gamma : [0,1] \times [0,1] \to G_2$ is continuous and $f \circ \Gamma(s,0) = f \circ \gamma_1(s) = f \circ (f^{-1} \circ \gamma_2)(s) = \gamma(s)$ for $s \in [0,1]$, $f \circ \Gamma(s,1) = f(c)$ for $s \in [0,1]$, and $f \circ \Gamma(0,1) = f \circ \Gamma(1,t)$ for $t \in [0,1]$. That is, $f \circ \Gamma$ is a path homotopy from γ_2 to constant $f(c)$ and so γ_2 is homotopic to zero. Since γ_2 is an arbitrary closed rectifiable curve in G_2, then G_2 is simply connected. □

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Lemma VII.4.3

Lemma VII.4.3. Let G be a region which is not the whole plane and such that every nonvanishing analytic function on G has an analytic square root. If $a \in G$ then there is an analytic function f on G such that:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one to one; and

(c) $f(G) = \{z \mid |z| < 1\}$.

Proof. Define $\mathcal{F}$ by letting

$$\mathcal{F} = \{f \in H(G) \mid f \text{ is one to one, } f(a) = 0, f'(a) > 0, f(G) \subset D\}.$$

For all $f \in \mathcal{F}$, since $f(G) \subset D$, then $\sup\{|f(z)| \mid z \in G\} \leq 1$. So $\mathcal{F}$ is locally bounded (by definition of "locally bounded") and so by Motel's theorem (Theorem VII.2.9) $\mathcal{F}$ is normal (if it is nonempty).

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Lemma VII.4.3 (continued 1)

Proof (continued). By hypothesis (since $z - b \neq 0$ on G), $z - b$ has an analytic square root on G, say $(g(z))^2 = z - b$. If $z_1, z_2 \in G$ an d$g(z_1) = \pm g(z_2)$ then $(g(z_1))^2 - z_1 - b = z_2 - b = (\pm g(z_2))^2$ and so $z_1 = z_2$. In particular, g is one to one. Since g is an analytic one to one function on G, by the Open Mapping Theorem (Theorem IV.7.5), $g(G)$ is open and so there if $r > 0$ such that $B(g(a); r) \subset g(G)$.

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Lemma VII.4.3 (continued 2)

Proof (continued). Let $g_1 = T \circ g$. Then g_1 is analytic and $g_1(G) \subset D$. If $\alpha = g_1(a)$ let $\varphi_\alpha - e^{i\theta_0}(z-\alpha)(\overline{\alpha}z-1)$. Notice

$$\varphi'_\alpha(z) - e^{i\theta_0}\frac{(\overline{\alpha}z-1)-(z-\alpha)\overline{\alpha}}{(\overline{\alpha}z-1)^2} = e^{i\theta_0}\frac{|\alpha|^2-1}{(\overline{\alpha}z-1)^2}.$$

Then φ_α is a Möbius transformation from D onto D such that $\varphi_\alpha(\alpha) = 0$ (this is shown in Exercise III.3.10). Define $g_2(z) = \varphi_\alpha \circ g_1(z)$. Then we still have $g_2(G) \subset D$ and g_2 is analytic on G, but we also have that $g_2(a) = 0$.

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$$g'_2(z)\Big|_{z=a} = [\varphi_\alpha(g_1(z))]'\Big|_{z=a} = \varphi'_\alpha(g_1(a))g'_1(a)$$

$$= \varphi'_\alpha(\alpha)g'_1(a) = e^{i\theta_0}\frac{1}{|\alpha|^2-1}g'_1(a).$$

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Proof (continued). Next, $g_1 = T \circ g$ where $T(z) = \frac{az+b}{cz+d}$, with $ad - bc \neq 0$, is some Möbius transformation. Now

$$T'(z) = \frac{1(cz+d) - (za+b)z}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0 \text{ for } z \neq \infty.$$

So $g_1'(a) = T'(g(z))g'(z)\Big|_{z=a} = T'(g(a))g'(a)$ where $T'(g(a)) \neq 0$.

Next, $(g(z))^2 = z - b$ and $2g(z)g'(z) = 1$, so $2g(z)g'(a) = 1$ and neither $g(a)$ nor $g'(a)$ equal 0. Hence $g_1(a) = T'(g(a))g'(a) \neq 0$ and $g_2'(a) = e^{i\theta_0}\frac{1}{|\alpha|^2-1}g - 1'(a) \neq 0$. So by the proper choice of θ_0 (namely, $-\arg(g_1'(a)/(|\alpha|^2-1))$) we have $g_2'(a) > 0$. So $g_2 \in \mathcal{F}$ and $\mathcal{F}$ is nonempty.

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(II) We now show that $\mathcal{F}^- = \mathcal{F} \cup \{0\}$ in $H(G)$. Suppose $\{f_n\}$ is a sequence in $\mathcal{F}$ and $f_n \to f$ in $H(G)$. Since $f_n(a) = 0$ for all $f_n \in \mathcal{F}$ then $f(a) = 0$. Also $f_n'(z) \to f'(a)$ so $f'(a) \geq 0$.

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Lemma VII.4.3 (continued 4)

Proof (continued). Let $z_1 \in G$, $\zeta = f(z_1)$, and $\zeta_n = f_n(z_1)$. Let $z_2 \in G$, $z_2 \neq z_1$ and let $K \subset G$ be a closed disk centered at z_2 such that $z_1 \notin K$. Since f_n is one to one and $f_n(z) - \zeta_n = 0$ at z_1 then $f_n(z) - \zeta_n$ does not vanish on K. But $f_n(z) - \zeta_n \to f(z) - \zeta$ uniformly on K (since K is compact), so by the corollary to Hurwitz's Theorem (Corollary VII.2.6) either $f(z) - \zeta$ never vanishes on K on $f(z) = \zeta$ on K. If $f(z) \equiv \zeta$ on K then f is constant throughout G (by Theorem IV.3.7) and since $f(a) = 0$ then $f(z) = 0$ on G.Otherwise $f(z) - \zeta$ never vanishes on K so (since $z_2 \in K$) $f(z_2) - \zeta \neq 0$ or $f(z_2) \neq \zeta = f(z_1)$. Since z_1 and z_2 are arbitrary distinct points in G, then g is one to one on G. By Exercise IV.7.4, $f'(z) \neq 0$ on G. We have above that $f'(a) \geq 0$, so it must be that in fact $f'(a) > 0$. So $f \in \mathcal{F}$.

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Proof (continued). Let $z_1 \in G$, $\zeta = f(z_1)$, and $\zeta_n = f_n(z_1)$. Let $z_2 \in G$, $z_2 \neq z_1$ and let $K \subset G$ be a closed disk centered at z_2 such that $z_1 \notin K$. Since f_n is one to one and $f_n(z) - \zeta_n = 0$ at z_1 then $f_n(z) - \zeta_n$ does not vanish on K. But $f_n(z) - \zeta_n \to f(z) - \zeta$ uniformly on K (since K is compact), so by the corollary to Hurwitz's Theorem (Corollary VII.2.6) either $f(z) - \zeta$ never vanishes on K on $f(z) = \zeta$ on K. If $f(z) \equiv \zeta$ on K then f is constant throughout G (by Theorem IV.3.7) and since $f(a) = 0$ then $f(z) = 0$ on G.Otherwise $f(z) - \zeta$ never vanishes on K so (since $z_2 \in K$) $f(z_2) - \zeta \neq 0$ or $f(z_2) \neq \zeta = f(z_1)$. Since z_1 and z_2 are arbitrary distinct points in G, then g is one to one on G. By Exercise IV.7.4, $f'(z) \neq 0$ on G. We have above that $f'(a) \geq 0$, so it must be that in fact $f'(a) > 0$. So $f \in \mathcal{F}$. That is, the limit of any sequence in $\mathcal{F}$ is either an element of $\mathcal{F}$ or the function $f(z) \equiv 0$ on G. Hence, $\mathcal{F}^- = \mathcal{F} \cup \{0\}$.

Lemma VII.4.3 (continued 4)

Proof (continued). Let $z_1 \in G$, $\zeta = f(z_1)$, and $\zeta_n = f_n(z_1)$. Let $z_2 \in G$, $z_2 \neq z_1$ and let $K \subset G$ be a closed disk centered at z_2 such that $z_1 \notin K$. Since f_n is one to one and $f_n(z) - \zeta_n = 0$ at z_1 then $f_n(z) - \zeta_n$ does not vanish on K. But $f_n(z) - \zeta_n \to f(z) - \zeta$ uniformly on K (since K is compact), so by the corollary to Hurwitz's Theorem (Corollary VII.2.6) either $f(z) - \zeta$ never vanishes on K on $f(z) = \zeta$ on K. If $f(z) \equiv \zeta$ on K then f is constant throughout G (by Theorem IV.3.7) and since $f(a) = 0$ then $f(z) = 0$ on G.Otherwise $f(z) - \zeta$ never vanishes on K so (since $z_2 \in K$) $f(z_2) - \zeta \neq 0$ or $f(z_2) \neq \zeta = f(z_1)$. Since z_1 and z_2 are arbitrary distinct points in G, then g is one to one on G. By Exercise IV.7.4, $f'(z) \neq 0$ on G. We have above that $f'(a) \geq 0$, so it must be that in fact $f'(a) > 0$. So $f \in \mathcal{F}$. That is, the limit of any sequence in $\mathcal{F}$ is either an element of $\mathcal{F}$ or the function $f(z) \equiv 0$ on G. Hence, $\mathcal{F}^- = \mathcal{F} \cup \{0\}$.

Lemma VII.4.3 (continued 5)

Proof (continued). (III) Finally, we show there is $f \in \mathcal{F}$ with $f(D) = D$. Consider the mapping of function $f \in H(G)$ to $f'(a) \in \mathbb{R}$. This is a continuous mapping by Theorem VII.2.1. Now since $\mathcal{F} \cup \{0\}$ maps G to D then $\mathcal{F} \cup \{0\}$ is locally bounded. By Part II, $\mathcal{F}^- = \mathcal{F} \cup \{0\}$ is closed so, by Corollary VII.2.10, $\mathcal{F}^-$ is compact. So by the Extreme Value Theorem (Corollary II.5.12) there is $f \in \mathcal{F}^-$ with $f'(a) \geq g'(a)$ for all $g \in \mathcal{F}$. Since $\mathcal{F}$ is nonempty by Part I, this particular f is in $\mathcal{F}$ (since $f \in \mathcal{F}$ implies $f(a) > 0$ and so $0(a) = 0$ does not give the maximum). We now show that for this $f \in \mathcal{F}$, we have $f(G) = D$ and this is the desired function.

Lemma VII.4.3 (continued 5)

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(IV) ASSUME $\omega \in D$ and $\omega \notin f(G)$. Then the function $(f(z) - \omega)/(1 - \overline{\omega} f(z))$ is analytic in G (notice $|1/\overline{\omega}| > 1$ so $f(z) \neq 1/\overline{\omega}$ for $z \in G$) and never vanishes in G.

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Lemma VII.4.3 (continued 6)

Proof (continued). Define $g : G \to \mathbb{C}$ as

$$g(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)}h(z)}.$$

Notice that from the definition of h,

$$2h(z)h'(z) = \frac{f'(z)(1 - \overline{\omega}f(z)) - (f(z) - \omega)(-\overline{\omega}f'(z))}{(1 - \overline{\omega}f(z))^2} = \frac{f'(z)(1 - |\omega|^2)}{(1 - \overline{\omega}f(z))^2}$$

and so

$$2h(a)h'(a) = \frac{f'(a)(1 - |\omega|^2)}{(1 - \overline{\omega}f(a))^2} = f'(a)(1 - |\omega|^2) \neq 0 \qquad (*)$$

since $\omega \in D$ and $f'(a) > 0$. Therefore $h'(a) \neq 0$ (and so $|h'(a)|/h'(a)$ is a complex number of modulus 1). Since $h(G) \subset D$ then $h(a) \in D$ and so $g(G) \subset D$. Also $g(a) = 0$ and g is one to one since h is (it is a branch of a square root and so is invertible, so h is one to one) and the Möbius transformation T such that $g(z) = T(h(z))$, is one to one.

Lemma VII.4.3 (continued 6)

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Lemma VII.4.3 (continued 7)

Proof (continued). Also,

$$g'(z) = \frac{|h'(a)|}{h'(a)} \frac{h'(a)(1-\overline{h(a)}h(z)) - (h(z)-h(a))(-\overline{h(a)}h'(z))}{(1-\overline{h(a)}h(z))^2}$$

$$= \frac{|h'(a)|}{h'(a)} \frac{h'(z)(1-|h(a)|^2)}{(1-\overline{h(a)}h(z))^2}$$

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$$g'(a) = \frac{|h'(a)|}{h'(a)} \frac{h'(a)(1-|h(a)|^2)}{(1-h(a)|^2)^2} = \frac{|h'(a)|}{1-|h(a)|^2}.$$

But $|h(a)|^2 = \left|\frac{f(a)-\omega}{1-\overline{\omega}f(a)}\right| = |-\omega| = |\omega|$ and by $(*)$, $2h'(a)h(a) = f'(a)(1-|\omega|^2)$, so ...

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Lemma VII.4.3 (continued 8)

Proof (continued).

$$g'(a) = \frac{|h'(a)|}{1-|h(a)|^2} = \frac{1}{1-|\omega|}\frac{|f'(a)|(1-|\omega|^2)}{2|h(a)|}$$

$$= \frac{f'(a)(1+|\omega|)2|h(a)| = \frac{f'(a)(1+|\omega|)}{2\sqrt{|\omega|}} = f'(a)\frac{1+|\omega|}{\sqrt{|\omega}} + \sqrt{|\omega|}}{>} f'(a)$$

because $2\sqrt{|\omega|} < 1 + |\omega|$ (since $0 < (1-|\omega|)^2 = 1 - 2|\omega| + |\omega|^2$). So g is one to one, $g(a) = 0$, $g(G) \subset D$, and $g'(a) > f'(a) > 0$. But the $g \in \mathcal{F}$ and $g'(a) > f'(a)$, a CONTRADICTION to the fact that $f'(a) \geq g'(a)$ for all $g \in \mathcal{F}$. So no such $\omega \in D$ with $\omega \notin f(G)$. That is, $f(G) = D$. Therefore, f is the desired function. □

Lemma VII.4.3 (continued 8)

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Theorem VII.4.2. The Riemann Mapping Theorem

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Let G be a simply connected region which is not the whole plane $\mathbb{C}$ and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$;
(b) f is one to one; and
(c) $f(G) = \{z \mid |z| < 1\}$.

Proof. Let G be a simply connected region which is not the whole plane $\mathbb{C}$. Notice that every nonvanishing analytic function on a simply connected region has an analytic square root on the region.

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Proof. Let G be a simply connected region which is not the whole plane $\mathbb{C}$. Notice that every nonvanishing analytic function on a simply connected region has an analytic square root on the region. This is because we can define a branch of the logarithm and compose it with such a function throughout the region; the branch cut must run from where the function is 0 to $\infty \in \mathbb{C}$; since the function can only be 0 outside of the simply connected region, then then a branch cut exists for the branch of the logarithm. The branch of the logarithm then allows us to define the square root (recall that a branch of the square root satisfies $z^{1/2} = e^{(1/2)\log z}$).

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Theorem VII.4.2. The Riemann Mapping Theorem (continued)

Proof. The existence of a function f satisfying (a), (b), and (c) now follows from Lemma VII.4.3. We need only prove uniqueness. Suppose g is another such function. Then $g^{-1} : D \to G$ is analytic by Corollary IV.7.6. Therefore $f \circ g^{-1} : D \to D$ is analytic, one to one, and onto. Also, $f \circ g^{-1}(0) = f(g^{-1}(0) = f(a) = 0$. So by Theorem VI.2.5, $f \circ g^{-1}(z) = c\varphi_0 = c(z - 0)/(1 - \overline{0}z) = cz$ (on D) for some $|c| = 1$. Replacing z with $g(z)$ in $f(g^{-1}(z)) = cz$ (since $g(G) = D$) gives $f(g^{-1}(g(z)) = cg(z)$ or $f(z) = cg(z)$. Then $0 < f'(z) - cg'(a)$; since $g'(a) > 0$ and $cg'(a) > 0$ then $c > 0$ and so $c = 1$.

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