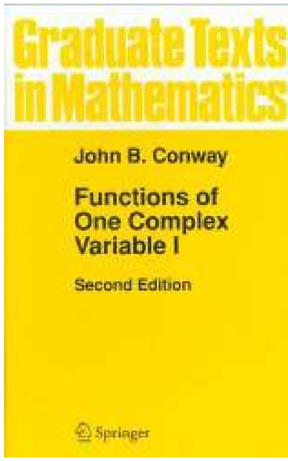
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.5. The Weierstrass Factorization Theorem—Proofs of Theorems



Lemma VII.5.A

Lemma VII.5.A. Let $\{z_n\}$ be a sequence of nonzero complex numbers. Suppose $\prod_{k=1}^{\infty} z_k$ exists. If $\prod_{k=1}^{\infty} a_k \neq 0$ then $\lim_{n\to\infty} z_n = 1$.

Proof. Denote $p_n = \prod_{k=1}^{n} z_k$. Suppose $\prod_{k=1}^{n} z_n$ exists and is not zero. Then no p_n is 0 and $p_n/p_{n-1} = z_n$. Since $\lim_{n\to\infty} p_n = z$, then

$$\lim_{n\to\infty} \frac{p_n}{p_{n-1}} = \lim_{n\to\infty} z_n \text{ implies } \frac{\lim_{n\to\infty} p_n}{\lim_{n\to\infty} p_{n-1}} = \lim_{n\to\infty} z_n,$$

or $1 = z/z = \lim_{n\to\infty} z_n$. □

Proposition VII.5.2

Proposition VII.5.2. Let $\mathrm{Re}(z) > 0$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proof. We just showed that if $\sum_{n=1}^{\infty} \log z_n$ converges (say to s) then $\prod_{k=1}^{\infty} z_n$ converges (to e^s). Now suppose $\prod_{n=1}^{\infty} z_n$ converges, say $\lim_{n\to\infty} p_n = z$ where $z = re^{i\theta}$ for some $-\pi < \theta \le \pi$. Define $\ell(p_n) = \log|p_n| + i\theta_n$ where $\theta - \pi < \theta_n \le \theta + \pi$. Since $\lim_{n\to\infty} p_n = z$ then $\lim_{n\to\infty} |p_n| = |z| = r$ and $\lim_{n\to\infty} \theta_n = \theta$; hence $\lim_{n\to\infty} \ell(p_n) = \lim_{n\to\infty}(\log|p_n| + i\theta_n) = \log|z| + i\theta$ (notice $\theta_n \in (\theta - \pi, \theta + \pi]$ for all $n \in \mathbb{N}$). If $s_n \sum_{k=1}^{n} \log z_k$ then $\exp(s_n) = p_n$ and so $s_n = \ell(p_n) + 2\pi i k_n$ for some $k \in \mathbb{Z}$. Since $p_n \to z$ then $s_n - s_{n-1} = \sum_{k=1}^{n} \log z_k - \sum_{k=1}^{n-1} \log z_k = \log z_n$ (where we use the principal branch of the logarithm here) and so $\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} z_n = \log \lim_{n\to\infty} z_n = \log 1 = 0$.

Proposition VII.5.2 (continued 1)

Proof (continued). Also

$$\ell(p_n) - \ell(p_{n-1}) = (\log|p_n| + i\theta_n) - (\log|p_{n-1}| + i\theta_{n-1})$$
$$= \log\left|\frac{p_n}{p_{n-1}}\right| + i(\theta_n - \theta_{n-1}) = \log|z_n| + i(\theta_n - \theta_{n-1})$$

and so

$$\lim_{n\to\infty}(\ell(p_n) - \ell(p_{n-1})) = \lim_{n\to\infty}(\log|z_n| + i(\theta_n - \theta_{n-1}))\log\left(\lim_{n\to\infty} z_n\right)$$
$$+i\lim_{n\to\infty}(\theta_n - \theta_{n-1}) = \log 1 + i(\theta - \theta) = 0.$$

Since $s_n = \ell(p_n) + 2\pi i k_n$ then $\ell(p_n) = s_n - 2\pi i k_n$, and so

$$\ell(p_n) - \ell(p_{n-1}) = (s_n - 2\pi i k_n) - (s_{n-1} - 2\pi i k_{n-1}) = s_n - s_{n-1} - 2\pi i(k_n - k_{n-1})$$

and $\lim_{n\to\infty}((s_n - s_{n-1}) - 2\pi i(k_n - k_{n-1})) = 0$, so $\lim_{n\to\infty}(k_n - k_{n-1}) = 0$. But since $k_n \in \mathbb{Z}$ then there is some $n_0 \in \mathbb{N}$ such that $k_m = k_n = k$ for some fixed $k \in \mathbb{Z}$ and for all $m, n \ge n_0$.

Proposition VII.5.2 (continued 2)

Proposition VII.5.2. Let $\text{Re}(z) > 0$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proof (continued). Therefore

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\ell(p_n) + 2\pi i k_n) = \lim_{n\to\infty} \ell(p_n) + 2\pi i \lim_{n\to\infty} k_n = \ell(z) + 2\pi i k.$$

That is, $\sum_{k=1}^{\infty} \log z_k$ converges. □

Lemma VII.5.B

Lemma VII.5.B. If $|z| < 1/2$ then $\frac{1}{2}|z| \le |\log(1+z)| \le \frac{3}{2}|z|$.

Proof. The power series for $\log 1 + z$ about $z = 0$ is

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

which has radius of convergence 1. So for $|z| < 1$.

$$\left|1 - \frac{\log(1+z)}{z}\right| = \left|\frac{1}{2}z - \frac{1}{3}z^2 + \frac{1}{4}z^3 - \cdots\right|$$

$$\le \frac{1}{2}|z| + \frac{1}{3}|z|^2 + \frac{1}{4}|z|^3 + \cdots \le \frac{1}{2}(|z| + |z|^2 + |z|^3 + \cdots) = \frac{1}{2}\frac{|z|}{1-|z|}.$$

For $|z| < 1/2$, $\left|1 - \frac{\log(1+z)}{z}\right| \le \frac{1}{2}$ and $|z - \log(1+z)| \le |z|/2$. So by Exercise I.3.1, $|\log(1+z)| - |z| \le |z|/2$ and so $|\log(1+z)| \le 3|z|/2$. Similarly, $|z| - |\log(1+z)| \le |z|/2$ and so $|z|/2 \le |\log(1+z)|$. □

Proposition VII.5.4

Proposition VII.5.4. Let $\text{Re}(z) > -1$. Then the series $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Proof. Suppose $\sum_{n=1} \infty z_n$ converges absolutely; that is, suppose $\sum_{n=1}^{\infty} |z_n|$ converges. Then, by the "Test for Divergence," from Calculus 2, $|z_n| \to 0$ and $z_n \to 0$. So there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $|z_n| < 1/2$. So by Lemma VII.5.B, for all $n \ge n_0$, $|log(1+z_n)| \le 3|z_n|/2$. So by the Direct Comparison Test, since $\sum_{n=1}^{\infty} 3|z_n|/2$ converges then $\sum_{n=1}^{\infty} |\log(1+z_n)|$ converges. That is, $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely.

Suppose $\sum_{n=1}^{\infty} |\log(1+z_n)|$ converges. Then by the Test for Divergence, $\lim_{n\to\infty} |\log(1+z_n)| = 0$ and so $\lim_{n\to\infty} z_n = 0$. so there is $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$ we have $|z_n| < 1/2$. By Lemma VII.5.B, for all $n \ge b_1$, $|z_n|/2 \le |\log(1+z_n)|$. By the Direct Comparison Test, since $\sum_{n=1}^{\infty} |\log(1+z_n)|$ converges then $\sum_{n=1}^{\infty} |z_n|/2$ converges. That is, $\sum_{n=1}^{\infty} z_n$ converges absolutely. □

Lemma VII.5.C

Lemma VII.5.C. Let $\{z_n\}$ be a sequence of complex numbers with $\text{Re}(z_n) > 0$ for all $n \in \mathbb{N}$ and suppose $\prod_{n=1}^{\infty} z_n$ converges absolutely. Then

(a) $\prod_{n=1}^{\infty} z_n$ converges; and

(b) any rearrangement of $\{z_n\}$, say $\{z_m\}$ (where $m = f(n)$ for some given one to one and onto $f : \mathbb{N} \to \mathbb{N}$) converges absolutely.

Proof. (a) Since $\prod_{n=1}^{\infty} z_n$ converges absolutely, by definition, the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely. By Proposition III.1.1, this means that $\sum_{n=1}^{\infty} \log z_n$ converges. So by Proposition VII.5.2, $\prod_{n=1}^{\infty} z_n$ converges.

(b) Since $\prod_{n=1}^{\infty} z_n$ converges absolutely then, by definition, the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely. That is, $\sum_{n=1}^{\infty} |\log z_n|$ converges. With $m = f(n)$ as described above (and $\{z_m\}$ a rearrangement of $\{z_n\}$), then by the Rearrangement Theorem from Calculus 2, $\sum_{n=1}^{\infty} |\log z_n| = \sum_{m=1}^{\infty} |\log z_m|$ and so $\sum_{m=1}^{\infty} \log z_m$ converges absolutely. So, by definition, $\prod_{m=1}^{\infty} z_m$ converges absolutely. □

Corollary VII.5.6

Corollary VII.5.6. If $\text{Re}(z_n) > 0$ then the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty}(z_n - 1)$ converges absolutely.

Proof. Suppose $\prod_{n=1}^{\infty} z_n$ converges absolutely. Then, by definition, $\sum_{n=1}^{\infty} \log z_n$ converges absolutely. Define $z_m = z_n - 1$. Then $\text{Re}(z_m) > -1$ and $\sum_{n=1}^{\infty} \log z_n = \sum_{m=1}^{\infty} \log(1 + z_m)$ converges absolutely. So by Proposition VII.5.4, $\sum_{m=1}^{\infty} z_m = \sum_{n=1}^{\infty}(z_n - 1)$ converges absolutely.

Suppose $\sum_{n=1}^{\infty}(z_n - 1)$ converges absolutely. Define $z_m = z_1 - 1$. Then $\text{Re}(z_m) > -1$ and $\sum_{n=1}^{\infty}(z_n - 1) = \sum_{m=1}^{\infty} z_m$ converges absolutely. So by Proposition VII.5.4, $\sum_{m=1}^{\infty} \log(1 + z_n) = \sum_{n=1}^{\infty} \log z_n$ converges absolutely. So, by definition, $\prod_{n=1}^{\infty} z_n$ converges absolutely. □

Lemma VII.5.7

Lemma VII.5.7. Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(z) \to f(z)$ uniformly for $x \in X$. If there is a constant a such that $\text{Re}(f(z)) \leq a$ for all $x \in X$, then $\exp(f_n(x)) \to \exp(f(x))$ uniformly for $x \in X$.

Proof. Let $\varepsilon > 0$. Since e^z is continuous at $z = 0$, there is $\delta > 0$ such that for $|z| < \delta$ such that for $|z| < \delta$ we have $|e^z - 1| < \varepsilon e^{-a}$. Choose $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|f_n(x) - f(x)| < \delta$ for all $x \in X$. Then for $n \geq n_0$ we have for all $x \in X$ that $\varepsilon e^{-a} > |e^{f_n(x) - f(x)} - 1| = |\exp(f_n(x))/\exp(f(x)) - 1|$.

Lemma VII.5.7 (continued)

Lemma VII.5.7. Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(z) \to f(z)$ uniformly for $x \in X$. If there is a constant a such that $\text{Re}(f(z)) \leq a$ for all $x \in X$, then $\exp(f_n(x)) \to \exp(f(x))$ uniformly for $x \in X$.

Proof (continued). So for all $n \geq n_0$ for all $x \in X$,

$$\begin{aligned} |\exp(f_n(x)) - \exp(f(x))| &< \varepsilon e^{-a} |\exp f(x)| \\ &= \varepsilon e^{-a} \exp(\text{Re}(f(x)) \\ &\qquad \text{since } |\exp(f(x))| = \exp(\text{Re}(f(x)) \\ &= \varepsilon \exp(\text{Re}(f(x)) - a) \\ &\leq \varepsilon \text{ since } \text{Re}(f(x)) - a \leq 0, \end{aligned}$$

That is, $\{\exp(f_n(x))\}$ converges to $\exp(f(x))$ uniformly on X. □

Lemma VII.5.8

Lemma VII.5.8. Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X to $\mathbb{C}$ such that $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely and uniformly for $x \in X$. Then the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for $x \in X$. Also, there is $n_0 \in \mathbb{N}$ such that $f(z) = 0$ if and only if $g_n(x) = -1$ for some n where $1 \leq n \leq n_0$.

Proof. The absolute and uniform convergence of $\sum_{n=1}^{\infty} g_n(x)$ on X implies that $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X for each $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty} |g_n(x)| < \varepsilon$ for all $x \in X$. In particular, there is $n_0 \in \mathbb{N}$ such that $|g_n(x)| < 1/2$ for all $x \in X$ and $n > n_0$. So for $n > n_0$, $\text{Re}(1 + g_n(x)) = \text{Re}(1) + \text{Re}(g_n(x)) > 1 - 1/2 = 1/2 > 0$, since $|\text{Re}(g_n(x))| \leq |g_n(x)| < 1/2$ and so $-1/2 < \text{Re}(g_n(x)) < 1/2$, for all $x \in X$. So by Lemma VII.5.B $|\log(1 + g_n(x))| \leq 3|g_n(x)|/2$ for $n > n_0$ and for all $x \in X$.

Lemma VII.5.8 (continued 1)

Proof (continued). Since $\sum_{n=1}^{\infty} 3|g_n(x)|/2$ converges uniformly for $x \in X$ then $h(x) = \sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))$ converges uniformly and absolutely for $x \in X$ (by a pointwise application of the Direct Comparison Test). Since each g_n is continuous then h is continuous by Theorem II.6.1. Since X is compact by hypothesis, then $h(X)$ is compact in $\mathbb{C}$ by Theorem II.5.8 and so h is bounded (since $h(X)$ is closed and bounded by the Heine-Borel Theorem). So there is some constant a such that $\text{Re}(h(x)) < a$ for all $x \in X$. So, by Theorem VII.5.7, $\exp h(x) = \prod_{n=n_0+1}^{\infty}(1 + g_n(x))$ converges uniformly for $x \in X$. Notice that since $\sum_{n=n_0+1}^{\infty} \log(1 + g_n(x))$ converges absolutely then, by definition, $\prod_{n=n_0+1}^{\infty}(1 + g_n(x))$ converges absolutely. Therefore,

$$f(x) = (1 + g_1(x))(1 + g_2(x)) \cdots (1 + g_{n_0}(x)) \exp(h(x)) = \prod_{n=1}^{\infty}(1 + g_n(x))$$

converges uniformly and absolutely for x in X, as claimed.

Lemma VII.5.8 (continued 2)

Lemma VII.5.8. Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X to $\mathbb{C}$ such that $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely and uniformly for $x \in X$. Then the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for $x \in X$. Also, there is $n_0 \in \mathbb{N}$ such that $f(z) = 0$ if and only if $g_n(x) = -1$ for some n where $1 \le n \le n_0$.

Proof (continued). Finally, since $\exp(h(x)) \neq 0$, then $f(x) = 0$ if and only if $1 + g_n(x) = 0$ for some $1 \le n \le n_0$; that is, if and only if $g_n(x) = -1$ for some $1 \le n \le n_0$. □

Theorem VII.5.9

Theorem VII.5.9. Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ (i.e., a sequence of analytic functions) such that no f_n is identically zero. If $\sum_{n=1}^{\infty}(f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of G, then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the function f_n at a.

Proof. Since $\sum_{n=1}^{\infty}(f_n(z) - 1)$ converges uniformly and absolutely on compact subsets of G (by hypothesis), then by Lemma VII.5.8, $f(z) = \prod_{n=1}^{\infty} f_n(z)$ converges uniformly and absolutely on compact subsets of G. Recall that uniform convergence on compact subsets of G implies convergence with respect to metric ρ on space $H(G)$ (see Proposition VII.1.10(b)). So the infinite product $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$.

Theorem VII.5.9 (continued)

Theorem VII.5.9. Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ (i.e., a sequence of analytic functions) such that no f_n is identically zero. If $\sum_{n=1}^{\infty}(f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of G, then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the function f_n at a.

Proof (continued). Let $a \in G$ be a zero of f. Choose $r > 0$ such that $\overline{B}(a; r) \subset G$. Since $\overline{B}(a; R) \subset G$ is compact, then $\sum_{n=1}^{\infty}(f_n(z) - 1)$ converges uniformly on $\overline{B}(a; r)$ by hypothesis. By Lemma VII.5.8 (see the proof) there is $n_0 \in \mathbb{N}$ such that $f(z) = f_1(z)f_2(z) \cdots f_n(z)g(z)$ where $g(z) \neq 0$ in $\overline{B}(a; r)$. So a is a zero of only n finite number of the functions f_n and the multiplicity of zero a of f is the sum of the multiplicities of a as a zero of the function f_n, as claimed. □

Lemma VII.5.11

Lemma VII.5.11. If $|z| \le 1$ and $p \ge 0$ then $|1 - E_p(z)| \le |z|^{p+1}$.

Proof. For $p = 0$, $|1 - E_0(z)| = |1 - (1 - z)| = |z| \le |z|^{p+1}$. For $p \ge 1$ fixed, $E_p(z)$ is analytic (entire, in fact) so $E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$ for some coefficients a_k ($E_p(0) = 1$, so $a_0 = 1$). Then from the definition of $E_p(z)$,

$$E_p'(z) = (-1)\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right)$$

$$+(1 - z)\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right)(1 + z + z^2 + \cdots + z^{p-1})$$

$$= (-1 + (1 - z^p))\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right)$$

$$= -z^p \exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right) \qquad (*)$$

Lemma VII.5.11 (continued 1)

Proof (continued). and from the power series representation

$$E_p'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}. \qquad (*)$$

We see from $(*)$ and $(**)$ that $a_1 = a_2 = \cdots = a_p = 0$. Now in the series expansion of $\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right)$ about $z = 0$, all coefficients are positive (since they are products and sums of exponential functions, which are 1 when evaluated at $z = 0$, and polynomials and their derivatives which are 0 when evaluated at $z = 0$), say

$$\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right) = 1 + \sum_{k=1}^{\infty} b_k z^k \text{ where } b_k > 0.$$

Lemma VII.5.11 (continued 2)

Proof (continued). So from $(*)$,

$$E_p'(z) = -2^p\left(1 + \sum_{k=1}^{\infty} b_k z^k\right) = -2^p - \sum_{k=1}^{\infty} b_k z^{k+p}$$

$$= \sum_{k=1}^{\infty} k a_k z^{k-1} \text{ by } (**)$$

and so $k a_k < 0$ for $k = p+1, p+2, \ldots$. Thus $|a_k| = -a_k$ for $k \ge p+1$. So for $z = 1$, $0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k$ since $a_1 = a_2 = \cdots = a = 0$, or $\sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1$. so for $|a| \le 1$,

$$|1 - E_p(z)| = |E_p(z) - 1| = \left|\left(1 + \sum_{k=p+1}^{\infty} a_k z^k\right) - 1\right|$$

$$= \left|\sum_{k=p+1}^{\infty} a_k z^k\right| = |z|^{p+1}\left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right| \ldots$$

Lemma VII.5.11 (continued 3)

Lemma VII.5.11. If $|z| \le 1$ and $p \ge 0$ then $|1 - E_p(z)| \le |z|^{p+1}$.

Proof (continued).

$$|1 - E_p(z)| \le |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k||z|^{k-p-1} \text{ by the Triangle Inequality and limits}$$

$$\le |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \text{ since } |z| \le 1$$

$$= |z|^{p+1} \text{ since } \sum_{k=p+1}^{\infty} |a_k| = 1,$$

and this is the claim. □

Theorem VII.5.12

Theorem VII.5.12. Let $\{a_n\}$ be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty}|z_n| = \infty$ and $a_z \neq 0$ for all $n \geq 1$. Suppose that no complex number is repeated in the sequence an infinite number of times. If $\{p_n\}$ is any sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

for all $r > 0$, then $f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a)$ converges in $H(\mathbb{C})$ (and so is analytic on $\mathbb{C}$). The function f is an entire function with zeros only at the points a_n If z_0 occurs in the sequence $\{a_n\}$ exactly n times then f has a zero at $z = z_0$ of multiplicity m. Furthermore, if $p_n = n-1$ then (5.13) will be satisfied.

Proof. Suppose integers $\{p_n\}$ exist such that (5.13) is satisfied.

Theorem VII.5.12 (continued 1)

Proof (continued). Then

$$\begin{aligned}\left|1 - E_{p_n}\left(\frac{z}{a}\right)\right| &\leq \left|\frac{z}{a}\right|^{p_n+1} \text{ by Lemma VII.5.11}\\ &\leq \left(\frac{r}{|a_n|}\right)^{p_n+1}\end{aligned}$$

for $|z| \leq r$ and for $r \leq |a_n|$ (so that $|z/a_n| \leq r/|a_n| \leq 1$). For a fixed $r > 0$ there is $N \in \mathbb{N}$ such that $|a_n| > r$ for all $n \geq N$ since $|a_n| \to \infty$. So for given $r > 0$ we have

$$\sum_{n=1}^{\infty}\left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \leq \sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} \text{ for } z \in \overline{B}(0;r).$$

By (5.13), the right hand side is finite and so $\sum_{n=1}^{\infty}\left(1 - E_{p_n}\left(\frac{z}{a_n}\right)\right)$ converges absolutely on $\overline{B}(0;r)$. So $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ converges in $H(G)$. Why does it converge uniformly?

Theorem VII.5.12 (continued 2)

Proof (continued). To show that $\{p_n\}$ can be found so that (5.13) holds for all r is easy; since $|a_n| \to \infty$ then "eventually" $|z_n| > r$ (for a given r) and we can take $p_n = n-1$ so that $\sum_{n=1}^{\infty}(r/|a_n|)^{p_n+1}$ can eventually be compared to a geometric series with ration less than 1. In particular, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n| > 2r$ and $r/|a_n| < 1/2$. then

$$\sum_{n=1}^{\infty}(r/|a_n|)^{p_n+1} = \sum_{n=1}^{\infty}(r/|a_n|)^n < \sum_{n=1}^{N}(r/|a_n|)^n + \sum_{n=N+1}^{\infty}(1/2)^n < \infty.$$

□

Theorem VII.5.14

Theorem VII.5.14. The Weierstrass Factorization Theorem. Let f be an entire function and let $\{a_n\}$ be the nonzero zeros of f repeated according to multiplicity. Suppose f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at 0 means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof. Since f is entire, by Theorem VII.5.12, there are nonnegative integers $\{p_n\}$ such that

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

has the same zeros as f with the same multiplicities. So $f(z)/h(z)$ has a removable singularities at $a = 0, a_1, a_2, \ldots$.

Theorem VII.5.14 (continued)

Theorem VII.5.14. The Weierstrass Factorization Theorem. Let f be an entire function and let $\{a_n\}$ be the nonzero zeros of f repeated according to multiplicity. Suppose f has a zero at $z=0$ of order $m \geq 0$ (a zero of order $m=0$ at 0 means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof (continued). Thus, f/h (reduced and the removable singularities removed) is nonzero then there is a branch of the logarithm defined on $(f/h)(\mathbb{C})$. So there is entire g such that $g(z) = \log(f(z)/h(z))$ or $f(z)/h(z) = e^{g(z)}$. Then

$$f(z) = h(z)e^{g(z)} = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

□

Theorem VII.5.15

Theorem VII.5.15. Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit points in G. Let $\{m_j\}$ be a sequence of nonnegative integers. Then there is an analytic function f defined on G whose only zeros are at the points a_j. Furthermore, a_j is a zero of f of multiplicity m_j.

Proof. (I) In Part I of the proof, we show that if the claim can be established for the special case where there is $R > 0$ such that

$$\{z \mid |z| > R\} \subset G \text{ and } |a_j| \leq R \text{ for all } j \geq 1, \qquad (5.16)$$

then the claim will hold. So hypothesize that f satisfying (5.16) exists with the added property that

$$\lim_{z\to\infty} f(z) = 1 \qquad (5.17)$$

and let G_1 be an arbitrary open set in $\mathbb{C}$ with $\{\alpha_j\}$ a sequence of distinct points in G_1 with no limit point and let $\{m_j\}$ be a sequence of integers.

Theorem VII.5.15 (continued 1)

Proof (continued). If $\overline{B}(a;r)$ is a disk in G, such that $\alpha_j \notin B(a;r)$ for all $j > 1$, consider the Möbius transformation $T(z) = (z-a)^{-1}$. Set $G = T(G_1) \setminus \{\infty\} \subset \mathbb{C}$. Then G satisfies (5.16) where $a_j = T(\alpha_j) = (\alpha_j - a)^{-1}$ since $\alpha_j \notin B(a;r)$ implies $a_j = T(\alpha_j) \in \overline{B}(a';R')$ for some $a' \in \mathbb{C}$, $' \in \mathbb{R}$, since T maps circles to circles (by Theorem III.3.14) and also $\mathbb{C} \setminus \overline{B}(a';R') \subset G$. If there is $f \in H(G)$ with a zero at each a_j of multiplicity m_j with no other zeros and such that f satisfies (5.17), then $g(z) = f(T(z))$ is analytic in $G_1 \setminus \{a\}$. Now

$$\begin{aligned} \lim_{z\to a} g(z) &= \lim_{z\to a} f(T(z)) \\ &= \lim_{z\to\infty} f(z) \text{ since } T(a) = \infty \\ &= 1 \text{ by (5.17)}, \end{aligned}$$

so g has a removable singularity at $z=a$. Furthermore, g has a zero at α_j of multiplicity m_j (since f has a zero at $a_j = T(\alpha_j)$ of multiplicity m_j). So g (with the removable discontinuity removed) is the desired function analytic on open set G_1.

Theorem VII.5.15 (continued 2)

Proof (continued). (II) Assume G satisfies (5.16). Define a sequence $\{z_n\}$ consisting of the points in $\{a_j\}$, but such that each a_j is repeated according to its multiplicity m_j. Since G is open and $\{z \mid |z| > R\} \subset G$ then $\mathbb{C} \setminus G$ is closed and bounded and so compact. So by Corollary II.5.14, for each $n \in \mathbb{N}$ there is $w_n \in \mathbb{C} \setminus G$ such that $|w_n - z_n| = d(z_n, \mathbb{C} \setminus G)$. Notice that condition (5.160 implies $|a_j| \leq R$ for all j, so if there are an infinite number of a_j'a then they must have a limit point by the Bolzano-Weierstrass Theorem (see `http://faculty.etsu.edu/gardnerr/4217/notes/2-3.pdf` for a statement in $\mathbb{R}$). since by hypothesis $\{a_j\}$ has no limit point in G so the limit point of $\{a_j\}$ is not in G and so $G \neq \mathbb{C}$. (If $\{a_j\}$ is finite, the result holds for a polynomial.) Now for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(z_n, \mathbb{C} \setminus G) < \varepsilon$, or else we could construct an infinite subsequence of $\{z_n\}$, say $\{z_{n'}\}$ is an infinite founded set since $|z_{n'}| \leq R$ for all $n' \in \mathbb{N}$ and so $z_{n'}\}$ has a limit point by the Bolzano-Weierstrass Theorem.

Theorem VII.5.15 (continued 3)

Proof (continued). But the limit point is not in $\mathbb{C} \setminus G$ by the condition $d(z_{n'}, \mathbb{C} \setminus G) \geq \varepsilon$, and so the limit point is in G, contradicting the hypothesis that $\{a_n\}$ has no limit point in G. So $\lim_{n\to\infty} d(z_n, \mathbb{C} \setminus G) = 0$ and hence $\lim_{n\to\infty} |z_n - w_n| = 0$. Consider the functions $E_n((z_n - w_n)/(z - w_n))$. Each has a simple zero at $z = z_n$ (where we take $(z_n - w_z)/z - w_n)$ to be 1 at $z = z_n$), and so the infinite product of the E_n's has the required zeros with the appropriate multiplicities. In Part III we show that the infinite product converges in $H(G)$.

(III) Let K be a compact subset in G. Then since both K and $\mathbb{C} \setminus G$ are compact, by Theorem II.5.17, $d(\mathbb{C} \setminus G, K) > 0$. For any $z \in K$ $d(w_n, K) \leq |z - w_n|$ and

$$\left|\frac{z_n - w_n}{z - w_n}\right| \leq |z_n - w_n|(d(w_n, K))^{-1} \leq |a_n - w_n|(d(\mathbb{C} \setminus G, K))^{-1}$$

since $w_n \in \mathbb{C} \setminus G$ and so $d(\mathbb{C} \setminus G, K) \leq d(w_n, K)$.

Theorem VII.5.15 (continued 4)

Proof (continued). As shown above, $\lim_{n\to\infty} |z_n - w_n| = 0$, so for any $0 < \delta < 1$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|(z_n - w_n)/(z - w_n)| < \delta$ for all $z \in K$. By Lemma VII.5.11, we have

$$\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| \leq \delta^{n+1} \qquad (5.18)$$

for all $n \geq N$ and $z \in K$. This gives (using the Direct Comparison Test and a geometric series with ration δ) that $\sum_{n=1}^{\infty}\left(E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right)$ converges absolutely and uniformly on K. By Theorem VII.5.9, $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z_n - w_n}{z - w_n}\right)$ converges in $H(G)$, so f is analytic on G. The second part of Theorem VII.5.9 implies that the points $\{a_j\}$ are the only zeros of f and m_j is the order of the zero at $z = a_j$ (because a_j occurs m_j times in the sequence $\{z_n\}$).

Theorem VII.5.15 (continued 5)

Proof (continued). To show (5.17) that $\lim_{z\to\infty} f(z) = 1$, let $\varepsilon > 0$ be an arbitrary number and let $R_1 > R$. If $|z| \geq R_1$ then, because $|z_n| \leq R$ and $w_n \in \mathbb{C} \setminus G \subset B(0; R)$, $\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{2R}{R - 1 - R}$. So if $R_1 > R$ satisfies $2R < \delta(R_1 - R)$ (that is, $R_1 > R + 2R/\delta$ and $\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{2R}{R_1 - R} < \delta$) for some $0 < \delta < 1$ then (5.18) holds for $|z| \geq R_1$ and for all $n \in \mathbb{N}$. In particular, $\mathrm{Re}\left(E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right) > 0$ for all $n \in \mathbb{N}$ and $|z| \geq R_1$ (for if this is less than or equal to 0, then $\left|\mathrm{Re}\left(\frac{z_n - w_n}{z - w_n}\right) = 1\right| \geq 1$ and (5.18) is violated).

Theorem VII.5.15 (continued 6)

Proof (continued). So

$$|f(z)-1| = \left|\prod_{n=1}^{\infty} E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| = \left|\exp\left(\sum_{n=1}^{\infty} \log E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right) = 1\right|$$

(5.19) is a "meaningful equation" (that is, $E_n((z_n - w_n)/(z - w_n)) \neq 0$ for $|z| \geq R_1$ and for $n \in \mathbb{N}$, and so there is a branch of the logarithm defined for all such $E_n((z_n - w_n)/(z - w_n))$, say the principal branch). Now we restrict $0 < \delta < 1/2$ so that (5.18) now gives for $|z| \geq R_1$ that $\left|F_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| \leq \left(\frac{1}{2}\right)^{n+1} \leq \frac{1}{2}$ for all $n \in \mathbb{N}$, and then by Lemma VII.5.B,

$$\log\left(E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right) = \log\left(\left(E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right) + 1\right) \leq \frac{3}{2}\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right|$$

for all $|z| \geq R_1$ and for all $n \in \mathbb{N}$.

Theorem VII.5.15 (continued 7)

Proof (continued). We now have

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\log\left(E_n\left(\frac{z_a-w_n}{z-w_n}\right)\right)\right| &\le \sum_{n=1}^{\infty}\left|\log E_n\left(\frac{z_n-w_n}{z-w_n}\right)\right| \\ &\le \sum_{n=1}^{\infty}\frac{3}{2}\left|E_n\left(\frac{z_n-w_n}{z-w_n}\right)-1\right| \\ &\le \sum_{n=1}^{\infty}\frac{3}{2}\delta^{n+1} \text{ by (5.18) (notice the choice of} \\ &\qquad R_1 \text{ implies that (5.18) holds for all } n\in\mathbb{N}) \\ &= \frac{3}{2}\frac{\delta^2}{1-\delta}\end{aligned}$$

for all $|z| \ge R_1$. By the continuity of e^z at $z = 0$, we can further restrict $0 < \delta < 1/2$ so that $|w| < \frac{3}{2}\frac{\delta^2}{1-\delta}$ implies $|e^w - 1| < \varepsilon$ (so that we now have δ "fixed").

Theorem VII.5.15 (continued 8)

Proof (continued). Then for $|z| \ge R_1$, equation (5.19) with our choice of δ (and with $w = \sum_{n=1}^{\infty}\log\left(E_n\left(\frac{z_n-w_n}{z-w_n}\right)\right)$) gives $|f(z)-1| = |e^w - 1| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary (R_1 is chosen based on δ and δ is chosen based on ε, so ultimately R_1 depends on ε), then $\lim_{z\to\infty} f(z) = 1$.

(IV) Combining Part III with Part II, gives an analytic function $f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z_n-w_n}{z-w_n}\right)$ which has a simple zero at $z = z_n$ for all $n \in \mathbb{N}$, and so has a zero at $z = a_j$ of multiplicity m_j for each $j \in \mathbb{N}$, on a set G satisfying (5.16) and such that $\lim_{z\to\infty} f(z) = 1$. By Part I, f can be modified to give the desired function g on any region G (in the proof of Part I the zeros of f are denoted as α_j instead of a_j). □

Corollary VII.5.20

Corollary VII.5.20. If f is a meromorphic function on an open set G then there are analytic functions g and h on G such that $f = g/h$.

Proof. Let $\{a_j\}$ be the poles of f and let m_j be the order of the pole at a_j. By Theorem VII.5.15, there is an analytic function h on G with a zero of multiplicity m_j at a_j for each $j \in \mathbb{N}$ and with not other zeros. So $h(z)f(z)$ has removable singularities at each point a_j, $j \in \mathbb{N}$. Setting $g = hf$ (reduced and removing the removable singularities), g is then analytic on G and $f = g/h$, as claimed. □