

Theorem VII.6.A

Theorem VII.6.A. For all $z \in \mathbb{C}$,

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

and the convergence is uniform over compact subset of \mathbb{C} .

Proof. The zeros of $\sin \pi z$ are precisely the integers, each of which is a simple zero because $[\sin \pi z]' = \pi \cos \pi z$ and $\pi \cos \pi \cdot 0 = \pi \neq 0$. For all $r > 0$

$$\sum_{n=-\infty}^{\infty} \left(\frac{r}{n}\right)^2 = 2 \sum_{n=1}^{\infty} \left(\frac{r}{n}\right)^2 = 2r^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(p series with $p = 2$), so with $p_n = 1$ for $n \in \mathbb{N}$, the hypothesis (5.13) of Theorem 5.12 is satisfied and we can use $\{p_n\}$ in the Weierstrass Factorization Theorem.

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Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.6. Factorization of the Sine Function—Proofs of Theorems



Theorem VII.6.A

Theorem VII.6.A (continued 1)

Proof (continued). So with $\{a_n\} = n$ for $n \in \mathbb{Z}$,

$$\sin \pi z = z e^{\mathcal{E}(z)} \prod_{n=-\infty}^{\infty} E_{p_n} \left(\frac{z}{a_n}\right) = z e^{\mathcal{E}(z)} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

for all $z \in \mathbb{C}$. Now the infinite product converges absolutely (see the proof of Theorem 5.12) and so the terms can be rearranged to give

$$\sin \pi z = z e^{\mathcal{E}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad (6.1)$$

since $(1 - z/n)(1 - z/(-n)) = 1 - z^2/n^2$ for all $n \in \mathbb{N}$. With $f(z) = \sin \pi z$ we have $f'(z) = \pi \cos \pi z$ and so $\pi \cot \pi z = \pi \cos \pi z / \sin \pi z = f'(z)/f(z)$.

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Theorem VII.6.A (continued 2)

Proof (continued). Now by Theorem VII.2.1,

$$\begin{aligned} \pi \cos \pi z = f'(z) &= [1] e^{\mathcal{E}(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) + z [e^{\mathcal{E}(z)} \mathcal{E}'(z)] \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \\ &\quad + z e^{\mathcal{E}(z)} \left(\sum_{j=1}^{\infty} \frac{-2z}{j^2} \prod_{n=1, n \neq j}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \right) \end{aligned}$$

and so

$$\begin{aligned} \pi \cos \pi z = \frac{f'(z)}{f(z)} &= \frac{1}{z} + \mathcal{G}'(z) + \sum_{j=1}^{\infty} \frac{-2z}{j^2} \frac{1}{(1 - z^2/j^2)} \\ &= \frac{1}{z} + \mathcal{G}'(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \end{aligned}$$

and by Exercise VIII.5.10 the convergence is uniform over compact subset of \mathbb{C} that contains no integers.

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Theorem VII.6.A (continued 3)

Proof (continued). By Exercise V.2.8, $\pi \cos \pi z + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ for z not an integer. So we have $g'(z) = 0$ and hence $g(z) = a$ for some constant $a \in \mathbb{C}$ on “appropriate sets.” Since $g(z)$ is entire, then $g(z) = z$ for all $z \in \mathbb{C}$. So from (6.1) for $z \neq 0$, $\frac{\sin \pi z}{\pi z} = \frac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ and

$$\lim_{z \rightarrow 0} \frac{\sin \pi z}{\pi z} = 1 = \lim_{z \rightarrow 0} \frac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{e^a}{\pi},$$

so that $e^a = e^g(z) = \pi$. Therefore

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

as claimed. The uniform convergence on compact sets claim follows from

Theorem VII.5.12. □

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