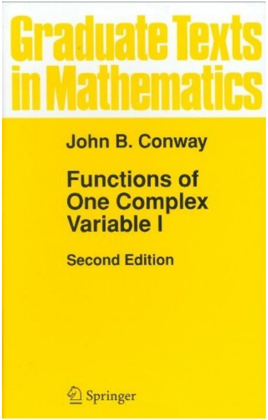
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.6. Factorization of the Sine Function—Proofs of Theorems



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and the convergence is uniform over compact subset of $\mathbb{C}$.

Proof. The zeros of $\sin \pi z$ are precisely the integers, each of which is a simple zero because $[\sin \pi z]' = \pi \cos \pi z$ and $\pi \cos \pi \cdot 0 = \pi \neq 0$. For all $r > 0$

$$\sum_{n=-\infty}^{\infty}{}' \left(\frac{r}{n}\right)^2 = 2 \sum_{n=1}^{\infty} \left(\frac{r}{n}\right)^2 = 2r^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(p series with $p = 2$), so with $p_n = 1$ for $n \in \mathbb{N}$, the hypothesis (5.13) of Theorem 5.12 is satisfied and we can use $\{p_n\}$ in the Weierstrass Factorization Theorem.

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Theorem VII.6.A (continued 1)

Proof (continued). So with $\{a_n\} = n$ for $n \in \mathbb{Z}$,

$$\sin \pi z = ze^{g(z)} \prod_{n=-\infty}^{\infty}{}' E_{p_n}\left(\frac{z}{a_n}\right) = ze^{g(z)} \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n}$$

for all $z \in \mathbb{C}$. Now the infinite product converges absolutely (see the proof of Theorem 5.12) and so the terms can be rearranged to give

$$\sin \pi z = ze^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad (6.1)$$

since $(1 - z/n)(1 - z/(-n)) = 1 - z^2/n^2$ for all $n \in \mathbb{N}$. With $f(z) = \sin \pi z$ we have $f'(z) = \pi \cos \pi z$ and so $\pi \cot \pi z = \pi \cos \pi z / \sin \pi z = f'(z)/f(z)$.

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Theorem VII.6.A (continued 2)

Proof (continued). Now by Theorem VII.2.1,

$$\pi\cos\pi z = f'(z) = [1]e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) + z[e^{g(z)}g'(z)]\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

$$+ze^{g(z)}\left(\sum_{j=1}^{\infty}\frac{-2z}{j^2}\prod_{n=1,n\neq j}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right)$$

and so

$$\pi\cos\pi z = \frac{f'(z)}{f(z)} = \frac{1}{z} + g'(z) + \sum_{j=1}^{\infty}\frac{-2z}{j^2}\frac{1}{(1-z^2/j^2)}$$

$$= \frac{1}{z} + g'(z) = \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

and by Exercise VII.5.10 the convergence us uniform over compact subset of $\mathbb{C}$ that contains no integers.

Theorem VII.6.A (continued 3)

Proof (continued). By Exercise V.2.8, $\pi \cos \pi z + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ for z not an integer. So we have $g'(z) = 0$ and hence $g(z) = a$ for some constant $a \in \mathbb{C}$ on "appropriate sets." Since $g(z)$ is entire, then $g(z) = z$ for all $z \in \mathbb{C}$. So from (6.1) for $z \neq 0$, $\dfrac{\sin \pi z}{\pi z} = \dfrac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)$ and

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so that $e^a = e^{g(z)} = \pi$.

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$$\sin\pi z = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right),$$

as claimed. The uniform convergence on compact sets claim follows from Theorem VII.5.12. □

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