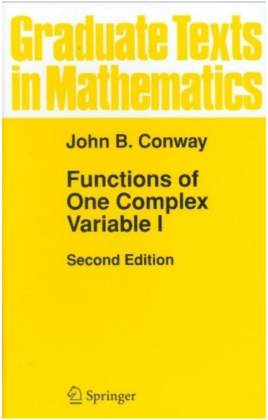
Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.7. The Gamma Function—Proofs of Theorems



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Lemma VII.7.A

Lemma VII.7.A. Let G be an open set in $\mathbb{C}$ and let $\{f_n\}$ be a sequence of analytic functions on G. Suppose $\{f_n\}$ converges to f (not identically 0) in $H(G)$. Then $\{f_n\}$ converges to f in $M(G)$ provided f is not the 0 function (the 0 function if not meromorphic since the zeros of a meromorphic function are isolated).

Proof. Let $\{f_n\}$ converge to f in $H(G)$. Then $f_n \to f$ in $C(G, \mathbb{C})$ (since $C(G, \mathbb{C})$ and $H(G)$ have the same metric). So $\lim_{n\to\infty} \rho(f_n, f) = 0$ where

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

and $\rho_n(f-g) = \max\{|f(z) - g(z)| \mid z \in K_n\}$ where K_n, $n \in \mathbb{N}$, are compact sets such that $G = \sup_{n=1}^{\infty} K_n$ and $K_n \subset \mathrm{int}(K_{n+1})$.

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$$d(z_1,z_2) = \frac{2|z_1-z_2|}{\{(1+|z_1|^2)(1+|z_2|)\}^{1/2}} \leq 2|z_1-z_2|.$$

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Lemma VII.7.A (continued)

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Proof (continued). Since an analytic functions does not take on the value ∞, then for the same compact K_n as above we have

$$2\max\{|f(z) = g(z)| \mid z \in K_n\} = 2\sup\{d(f(z), g(z)) \mid z \in K_n\}$$

and so $\rho_\infty(f, g) \leq 2\rho(f, g)$ (where "ρ_∞" denotes the metric on $C(G, \mathbb{C}_\infty)$). So $\lim_{n\to\infty} \rho(f_n, f) = 0$ implies $\lim_{n\to\infty} \rho_\infty(f_n, f) = 0$. □

Theorem VII.7.B

Theorem VII.7.B. Gauss's Formula.
For $z \neq 0, -1, -2, \ldots$ we have

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

Proof. We have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

$$= \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{n=1}^{n} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} = \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k}{z+k} e^{z/k}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z}}{z} \frac{1}{z+1} \frac{2}{z+2} \frac{3}{z+3} \cdots \frac{n}{z+n} e^{z/1} e^{z/2} e^{z/3} \cdots e^{z/n}$$

$$= \lim_{n\to\infty} \frac{e^{-\gamma z} n!}{z(z+1)(z+2)\cdots(z+3)} \exp\left(z\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right). \quad (*)$$

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Theorem VII.7.B (continued 1)

Proof (continued). But

$$e^{-\gamma z} \exp\left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) z\right)$$

$$= \exp\left(-\gamma z + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) z\right)$$

$$= \exp(z \log n) \exp(-z \log n) \exp\left(z\left(-\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right)$$

$$= \exp(z \log n) \exp\left(z\left(-\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log z\right)\right)$$

$$= n^z \exp\left(z\left(-\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log z\right)\right).$$

Theorem VII.7.B (continued 2)

Proof (continued). So

$$\lim_{n\to\infty} e^{-\gamma z} \exp\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) z\right)$$

$$= \lim_{n\to\infty} n^z \exp\left(z\left(-\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log z\right)\right) = \lim_{n\to\infty} n^z$$

since $\lim_{n\to\infty} \left(z\left(-\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log z\right)\right) = z(-\gamma+\gamma) = 0.$

So $(*)$ becomes

$$\Gamma(z) = \lim_{n\to\infty} \frac{e^{-\gamma z} n!}{z(z+1)(z+2)\cdots(z+n)} \exp\left(z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty} n! n^z z(z+1)(z+2)\cdots(z+n).$$

□

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Theorem VII.6.C

Theorem VII.7.C. Functional Equation.
For $z \neq 0, -1, -2, \ldots$, $\Gamma(z+1) = \Gamma(z)$.

Proof. By Gauss's Formula,

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)\cdots(z+n+1)}$$

$$= \lim_{n\to\infty} z\left(\frac{n!n^{z}}{z(z+1)(z+2)\cdots(z+n)}\right)\left(\frac{n}{z+n+1}\right) = z\Gamma(z)(1) = z\Gamma(z).$$

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Lemma VII.7.D

Lemma VII.7.D. The residue of the gamma function Γ at simple pole $-n$, $n \in \mathbb{N} \cup \{0\}$, is $\text{Res}(\Gamma; -n) = (-1)^n/n!$.

Proof. By Proposition V.2.4 (with $m = 1$), $\text{Res}(\Gamma; -n) = \lim_{z \to -n}(z+n)\Gamma(z)$ (see the note in the class notes following the statement of Proposition V.2.4). By Theorem VII.7.C, $\Gamma(z+n+1) = z(z+1)(z+2)\cdots(z+n-1)(z+n)\Gamma(z)$ and

$$(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n-1)}.$$

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So

$$\text{Res}(\Gamma; -n) = \lim_{z \to -n}(z + n)\Gamma(z) = \lim_{z \to -n} \frac{\Gamma(z + n + 1)}{z(z + 1)(z + 2) \cdots (z + n - 1)}$$

$$= \frac{\Gamma(1)}{(-n(-(n - 1))(-(z - 2)) \cdots (-2)(-1)} = \frac{(1)}{(-1)^n n!} \frac{(-1)^n}{n!}.$$

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Lemma VII.7.E

Lemma VII.7.E. $\log \Gamma(x)$ is a convex function for $x > 0$.

Proof. Notice from the definition of Γ that $\Gamma(x) > 0$ for $x > 0$. By Exercise VII.5.10 we have

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

for $z \neq 0, -1, -2, \ldots$ and convergence is uniform on every compact subset of $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

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$$\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)' = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{n(n+2) - nz}{(n(n+z))^2} = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2}$$

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Lemma VII.7.E (continued)

Lemma VII.7.E. $\log \Gamma(x)$ is a convex function for $x > 0$.

Proof (continued). Now

$$\frac{d^2}{dz^2}[\log \Gamma(z)] = \frac{d}{dz}\left[\frac{1}{\Gamma(z)}\Gamma'(z)\right] = \left(\frac{\Gamma'(z)}{\Gamma(z)}\right)'$$

and so for $x > 0$

$$\frac{d^2}{dz^2}[\log \Gamma(z)\Big|_{z=x} = \left(\frac{\Gamma'(z)}{\Gamma(z)}\right)'\Big|_{z=x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} > 0$$

and so by Proposition VI.3.4, $\log \Gamma(x)$ is convex. □

Theorem VII.7.13

Theorem VII.7.13. Bohr-Mollerup Theorem.
Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties:

(a) $\log f(x)$ is a convex function;
(b) $f(x+1) = xf(x)$ for all $x > 0$;
(c) $f(1) = 1$.

Then $f(x) = \Gamma(x)$ for $x > 0$.

Proof. By (b), for all $n \in \mathbb{N}$ we have

$$f(x+n) = (x+n-1)f(x+n-1) = (x+n-2)(x+n-1)f(x+n-2) = \cdots = x(x+$$

Notice that for $n = 0$ this reduces to $f(x) = f(x)$. So if $f(x) = \Gamma(x)$ for $0 < x \leq 1$ then (7.14) implies $f(x) = \Gamma(x)$ for all $x > 0$.

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$$\frac{\log f(n-1) - \log f(n)}{(n-1) - n} \leq \frac{\log f(x+n) - \log f(n)}{(x+n) - n} \leq \frac{\log f(n+1) - \log f(n)}{(n+1) - n}.$$

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Theorem VII.7.13 (continued 1)

Proof (continued). By (7.14) and (c) we have that $f(m) = (m-1)!$ for $m \in \mathbb{N}$. Then we have for $n \in \mathbb{N}$, $n > 2$,

$$-\log(n-2)! + \log(n-1)! \le \frac{\log f(x+n) - \log(n-1)!}{x} \le \log n! - \log(n-1)!$$

or

$$\log(n-1) \le \frac{\log f(x+n) - \log(n-1)!}{x} \le \log n$$

or

$$x \log(n-1) \le \log f(x+n) - \log(n-1)! \le x \log n.$$

So $x\log(n-1) + \log(n-1)! \le \log f(x+n) \le x \log n + \log(n-1)!$ and exponentiating

$$\exp(x\log(n-1) + \log(n-1)!) \le \exp(\log f(x+n)) \le \exp(x \log n + \log(n-1)!)$$

or $(n-1)^x(n-1)! \le f(x+n) \le n^x(n-1)!$. By (7.14)
$(n-1)^x(n-1)! \le x(x+1)\cdots(x+n-1)f(x) \le n^x(n-1)!$ or

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$$x\log(n-1) \le \log f(x+n) - \log(n-1)! \le x\log n.$$

So $x\log(n-1) + \log(n-1)! \le \log f(x+n) \le x\log n + \log(n-1)!$ and exponentiating

$$\exp(x\log(n-1) + \log(n-1)!) \le \exp(\log f(x+n)) \le \exp(x\log n + \log(n-1)!)$$

or $(n-1)^x(n-1)! \le f(x+n) \le n^x(n-1)!$. By (7.14) $(n-1)^x(n-1)! \le x(x+1)\cdots(x+n-1)f(x) \le n^x(n-1)!$ or

Theorem VII.7.13 (continued 2)

Proof (continued).

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \le f(x) \le \frac{n^x(n-1)!}{x(x+1)\cdots(x+n-1)}$$

$$= \frac{n^x n!}{x(x+1)\cdots(x+n)}\frac{x+n}{n}.$$

Now the left hand side of this equation is independent of the right side so that we can conclude

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \le f(x) \le \frac{m^x m!}{x(x+1)\cdots(x+m)}\frac{x+m}{m}$$

for all $n, m \in \mathbb{N}$, $n > 2$, $m > 2$.

Theorem VII.7.13 (continued 2)

Proof (continued).

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \leq f(x) \leq \frac{n^x(n-1)!}{x(x+1)\cdots(x+n-1)}$$
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for all $n, m \in \mathbb{N}$, $n > 2$, $m > 2$. So with n replaced with $n+1$ on the left hand side and m replaced with n on the right hand side we have

$$\frac{n^x n!}{x(x-1)\cdots(x+n)} \leq f(x) \leq \frac{n^x n!}{x(x_1)\cdots(x+n)}\frac{x+n}{n}$$

for $n > 2$ and $x \in (0, 1]$.

Theorem VII.7.13 (continued 2)

Proof (continued).

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \le f(x) \le \frac{n^x(n-1)!}{x(x+1)\cdots(x+n-1)}$$
$$= \frac{n^x n!}{x(x+1)\cdots(x+n)}\frac{x+n}{n}.$$

Now the left hand side of this equation is independent of the right side so that we can conclude

$$\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)} \le f(x) \le \frac{m^x m!}{x(x+1)\cdots(x+m)}\frac{x+m}{m}$$

for all $n, m \in \mathbb{N}$, $n > 2$, $m > 2$. So with n replaced with $n+1$ on the left hand side and m replaced with n on the right hand side we have

$$\frac{n^x n!}{x(x-1)\cdots(x+n)} \le f(x) \le \frac{n^x n!}{x(x_1)\cdots(x+n)}\frac{x+n}{n}$$

for $n > 2$ and $x \in (0, 1]$.

Theorem VII.7.13 (continued 3)

Proof (continued). Since $\lim_{n\to\infty}(x+n)/n = 1$ for $x \in (0,1]$, then the inequality implies that $f(x) = \dfrac{n^x n!}{(x(x+1)\cdots(x+n)}$ for $x \in (0,1)$ an d$n > 2$. So by Gauss's Formula (Theorem VII.7.B) $f(x) = \Gamma(x)$ for $x \in (0,1]$. For $x' > 1$, say $x' = x + n$ where $n \in \mathbb{N}$, (7.14) gives

$$f(x') = f(x+n) = x(x+1)\cdots(x+n-1)f(x) = x(x+1)\cdots(x+n-1)\Gamma(x).$$

By the Functional Equation (Theorem VII.7.C),

$$\Gamma(x') = \Gamma(x+n) = x(x+1)\cdots(x+n-1)f(x) = f(x').$$

So $f(x) = \Gamma(x)$ for $x > 0$, as claimed. □

Theorem VII.7.13 (continued 3)

Proof (continued). Since $\lim_{n\to\infty}(x+n)/n = 1$ for $x \in (0,1]$, then the inequality implies that $f(x) = \dfrac{n^x n!}{(x(x+1)\cdots(x+n)}$ for $x \in (0,1)$ an d$n > 2$. So by Gauss's Formula (Theorem VII.7.B) $f(x) = \Gamma(x)$ for $x \in (0,1]$. For $x' > 1$, say $x' = x + n$ where $n \in \mathbb{N}$, (7.14) gives

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By the Functional Equation (Theorem VII.7.C),

$$\Gamma(x') = \Gamma(x+n) = x(x+1)\cdots(x+n-1)f(x) = f(x').$$

So $f(x) = \Gamma(x)$ for $x > 0$, as claimed. □

Lemma VII.7.16

Lemma VII.7.16. Let $S = \{z \mid a \leq \text{Re}(z) \leq A\}$ where $0 < a < A < \infty$.

(a) For every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in S$,
$$\left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| < \varepsilon \text{ whenever } 0 < \alpha < \beta < \delta.$$

(b) For every $\varepsilon > 0$ there is a number κ such that for all $z \in S$,
$$\left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| < \varepsilon \text{ whenever } \beta > \alpha > \kappa.$$

Proof. (a) For $0 < t \leq 1$, $\log t \leq 0$ and so for $z \in S = \{z \mid z \leq \text{Re}(z) \leq A\}$ we have $\text{Re}(z) - 1 \geq z - 1$ and so $(\text{re}(z) - 1) \log t \leq (a - 1) \log t$. Since $e^{-t} < 1$, then
$$|e^{-t} t^{z-1}| \leq |t^{z-1}| = |e^{(z-1)\log t}| = e^{\text{Re}(z-1)\log t} = t^{\text{Re}(z-1)} \leq t^{a-1}.$$

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$$|e^{-t} t^{z-1}| \leq |t^{z-1}| = |e^{(z-1)\log t}| = e^{\text{Re}(z-1)\log t} = t^{\text{Re}(z-1)} \leq t^{a-1}.$$

So for $0 < \alpha < \beta < 1$, we have for all $z \in S$ that

$$\left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| \leq \int_\alpha^\beta |e^{-t} t^{z-1}|\, dt \leq \int_\alpha^\beta t^{a-1}\, dt = \frac{1}{z}(\beta^a - \alpha^a).$$

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So for $0 < \alpha < \beta < 1$, we have for all $z \in S$ that

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Lemma VII.7.16 (continued)

Proof (continued). If $\varepsilon > 0$, there is $0 < \delta < 1$ such that $(\beta^a - \alpha^a)/a < \varepsilon$ for $|\alpha - \beta| < \delta$ (since $f(x) = x^a/a$ is uniformly continuous on $[0, 1]$), as claimed.

Lemma VII.7.16 (continued)

Proof (continued). If $\varepsilon > 0$, there is $0 < \delta < 1$ such that $(\beta^a - \alpha^a)/a < \varepsilon$ for $|\alpha - \beta| < \delta$ (since $f(x) = x^a/a$ is uniformly continuous on $[0, 1]$), as claimed.

(b) For $z \in S = \{z \mid a \leq \text{Re}(z) \leq A\}$ and $t \geq 1$, $|t^{z-1}| = |e^{(z-1)\vee t}| = e^{\text{Re}(z-1)\log t} = t^{z-1} \leq t^{A-1}$. Since $t^{A-1}e^{-t/2}$ is continuous on $[1, \infty)$ and converges to zero as $t \to \infty$, there is constant c such that $t^{A-1}e^{-t/2} \leq c$ for all $t \geq 1$.

Lemma VII.7.16 (continued)

Proof (continued). If $\varepsilon > 0$, there is $0 < \delta < 1$ such that $(\beta^a - \alpha^a)/a < \varepsilon$ for $|\alpha - \beta| < \delta$ (since $f(x) = x^a/a$ is uniformly continuous on $[0,1]$), as claimed.

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$$\left|\int_\alpha^\beta e^{-t}t^{z-1}\,dt\right| \leq \int_\alpha^\beta |e^{-t}e^{z-1}|\,dt \leq c\int_\alpha^\beta e^{-t/2}\,dt = 2c(e^{-\alpha/2} - e^{-\beta/2}).$$

Lemma VII.7.16 (continued)

Proof (continued). If $\varepsilon > 0$, there is $0 < \delta < 1$ such that $(\beta^a - \alpha^a)/a < \varepsilon$ for $|\alpha - \beta| < \delta$ (since $f(x) = x^a/a$ is uniformly continuous on $[0,1]$), as claimed.

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If $\varepsilon > 0$, there is $\kappa > 1$ such that $|2c(e^{-\alpha/2} - e^{-\beta/2}| < \varepsilon$ whenever $\alpha, \beta > \kappa$ (since $\lim_{t\to\infty} e^{-t/2} = 0$), as claimed. □

Lemma VII.7.16 (continued)

Proof (continued). If $\varepsilon > 0$, there is $0 < \delta < 1$ such that $(\beta^a - \alpha^a)/a < \varepsilon$ for $|\alpha - \beta| < \delta$ (since $f(x) = x^a/a$ is uniformly continuous on $[0,1]$), as claimed.

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Proposition VII.7.17

Proposition VII.7.17. If $G = \{z \mid \mathrm{Re}(z) < 0\}$ and

$$f_n(z) = \int_{1/n}^{n} e^{-t} t^{z-1}\, dt$$

for $n \in \mathbb{N}$ and $z \in G$, then each f_n is analytic on G and the sequence is convergent on $H(G)$.

Proof. With $\varphi(t,z) = e^{-t}t^{z-1}$ and $\gamma = [1/n, n]$, $\int_\gamma \varphi(t,z)\, dt = \int_{1/n}^{n} e^{-t}t^{z-1}\, dt = f_n(z)$ is analytic on G by Exercise IV.2.2. If K is a compact subset of G then K is closed and bounded (by the Heine-Borel Theorem) and so $K \subset \{z \mid z \le \mathrm{Re}(z) \le A\}$ for some $a, A \in \mathbb{R}$.

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$$f_m(z) - f_n(z) = \int_{1/m}^{m} e^{-t} t^{z-1}\, dt + \int_{1/n}^{n} e^{-t} t^{z-1}\, dt$$

$$= \int_{z/m}^{1/n} e^{-t} t^{z-1}\, dt \int_{n}^{m} e^{-t} t^{z-1}\, dt,$$

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Proof. With $\varphi(t,z) = e^{-t} t^{z-1}$ and $\gamma = [1/n, n]$, $\int_\gamma \varphi(t,z)\, dt = \int_{1/n}^{n} e^{-t} t^{z-1} \, dt = f_n(z)$ is analytic on G by Exercise IV.2.2. If K is a compact subset of G then K is closed and bounded (by the Heine-Borel Theorem) and so $K \subset \{z \mid z \le \text{Re}(z) \le A\}$ for some $a, A \in \mathbb{R}$. Since for $m > n$,

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Proposition VII.7.17 (continued)

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for $n \in \mathbb{N}$ and $z \in G$, then each f_n is analytic on G and the sequence is convergent on $H(G)$.

Proof (continued). then by Lemma VII.7.16, for all $\varepsilon > 0$, if m, n are sufficiently large ($0 < 1/m < 1/n < \delta$ and $m > n > \kappa$ in the notation of Lemma VII.7.16) we have $|f_m(z) - f_n(z)| < \varepsilon$ for all $a \in K$. Since K is an arbitrary compact subset of G and $\varepsilon > 0$ is arbitrary, then by Lemma VII.1.7(a), for m, n sufficiently large, $\rho(f_m, f_n) < \varepsilon$ for a given $\varepsilon > 0$. That is, $\{f_n\}$ is a Cauchy sequence in $H(G)$. Since $H(G)$ is complete by Corollary VII.2.3, then $\{f_n\}$ converges in $H(G)$. □

Proposition VII.7.17 (continued)

Proposition VII.7.17. If $G = \{z \mid \text{Re}(z) < 0\}$ and

$$f_n(z) = \int_{1/n}^{n} e^{-t} t^{z-1}\, dt$$

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Proof (continued). then by Lemma VII.7.16, for all $\varepsilon > 0$, if m, n are sufficiently large ($0 < 1/m < 1/n < \delta$ and $m > n > \kappa$ in the notation of Lemma VII.7.16) we have $|f_m(z) - f_n(z)| < \varepsilon$ for all $a \in K$. Since K is an arbitrary compact subset of G and $\varepsilon > 0$ is arbitrary, then by Lemma VII.1.7(a), for m, n sufficiently large, $\rho(f_m, f_n) < \varepsilon$ for a given $\varepsilon > 0$. That is, $\{f_n\}$ is a Cauchy sequence in $H(G)$. Since $H(G)$ is complete by Corollary VII.2.3, then $\{f_n\}$ converges in $H(G)$. □

Lemma VII.7.19

Lemma VII.7.19.

(a) The sequence $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$.

(b) If $t \geq 0$ then $(1-t/n)^n \leq e^{-1}$ for all $n \geq t$.

Proof. (a) Let H be a compact subset of $\mathbb{C}$. Then K is closed and bounded so $|z| < n$ for all $z \in K$ and n sufficiently large. If we show $\lim_{n\to\infty} n\log(1+z/n) = z$ uniformly for $z \in K$, then by Lemma VII.5.7 $(1+z/n)^n \to e^z$ uniformly on K and, since K is an arbitrary compact set in $\mathbb{C}$, then by Proposition VII.1.10(b) $(1+z/n)^n \to e^z$ in $H(\mathbb{C})$.

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Recall that $\log(1+w) = \sum_{k=1}^{\infty}(-1)^{k-1}w^k/k$ for $|w| < 1$. Let $n > |z|$ for all $z \in K$. If $z \in K$ then $n\log\left(1+\frac{z}{n}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{kn^{k-1}}$ or

$$n\log\left(1+\frac{z}{n}\right) - z = z\sum_{k=2}^{\infty}(-1)^{k-1}\frac{z^{k-1}}{kn^{k-1}}. \qquad (7.20)$$

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Proof. (a) Let H be a compact subset of $\mathbb{C}$. Then K is closed and bounded so $|z| < n$ for all $z \in K$ and n sufficiently large. If we show $\lim_{n\to\infty} n \log(1+z/n) = z$ uniformly for $z \in K$, then by Lemma VII.5.7 $(1+z/n)^n \to e^z$ uniformly on K and, since K is an arbitrary compact set in $\mathbb{C}$, then by Proposition VII.1.10(b) $(1+z/n)^n \to e^z$ in $H(\mathbb{C})$.

Recall that $\log(1+w) = \sum_{k=1}^\infty (-1)^{k-1} w^k/k$ for $|w| < 1$. Let $n > |z|$ for all $z \in K$. If $z \in K$ then $n \log\left(1+\frac{z}{n}\right) = \sum_{k=1}^\infty (-1)^{k-1} \frac{z^k}{kn^{k-1}}$ or

$$n \log\left(1+\frac{z}{n}\right) - z = z \sum_{k=2}^\infty (-1)^{k-1} \frac{z^{k-1}}{kn^{k-1}}. \qquad (7.20)$$

Lemma VII.7.19 (continued 1)

Proof (continued). So

$$\left| n \log\left(1 + \frac{z}{n}\right) - z \right| \leq |z| \sum_{k=2}^{\infty} \frac{1}{k} \left|\frac{z}{n}\right|^{k-1} \leq |z| \sum_{k=2}^{\infty} \left|\frac{z}{n}\right|^{k-1}$$

$$= |z| \sum_{k=1}^{\infty} \left|\frac{z}{k}\right|^{k} = |z| \frac{|z/n|}{1 - |z/n|} = \frac{|z|^2}{n} \frac{1}{1 - |z/n|} = \frac{|z|^2}{n - |z|} \leq \frac{R^2}{n - R}$$

where $R \geq |z|$ for all $z \in K$. If $n \to \infty$ then the (uniform) bound $R^2/(n - R)$ gives to 0 and so $n \log\left(1 + \frac{z}{n}\right) \to z$ uniformly and, as described above, the result now follows.

Lemma VII.7.19 (continued 1)

Proof (continued). So

$$\left| n \log \left(1 + \frac{z}{n}\right) - z \right| \le |z| \sum_{k=2}^{\infty} \frac{1}{k} \left|\frac{z}{n}\right|^{k-1} \le |z| \sum_{k=2}^{\infty} \left|\frac{z}{n}\right|^{k-1}$$

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Lemma VII.7.19 (continued 2)

Lemma VII.7.19.

(a) The sequence $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$.

(b) If $t \geq 0$ then $(1-t/n)^n \leq e^{-1}$ for all $n \geq t$.

Proof (continued). (b) Let $t \geq 0$ where $t \leq n$ (so $0 \leq t \leq n$) and substitute $-t$ for z in (7.20) (so $|-t| - t \leq n$ and (7.20) holds for all $|z| \leq n$). This gives

$$n \log\left(1-\frac{t}{n}\right) + t = -\left(t\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{t}{n}\right)^{k-1}\right) \leq 0.$$

Thus $n\log(1-t/n) \leq -t$ and exponentiating $e^{n\log(1-t/n)} \leq e^{-t}$ or $(1-t/n)^n \leq e^{-t}$ where $n \geq t$, as claimed. □

Lemma VII.7.19 (continued 2)

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(a) The sequence $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$.

(b) If $t \geq 0$ then $(1-t/n)^n \leq e^{-1}$ for all $n \geq t$.

Proof (continued). (b) Let $t \geq 0$ where $t \leq n$ (so $0 \leq t \leq n$) and substitute $-t$ for z in (7.20) (so $|-t|-t \leq n$ and (7.20) holds for all $|z| \leq n$). This gives

$$n\log\left(1-\frac{t}{n}\right)+t=-\left(t\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{t}{n}\right)^{k-1}\right) \leq 0.$$

Thus $n\log(1-t/n) \leq -t$ and exponentiating $e^{n\log(1-t/n)} \leq e^{-t}$ or $(1-t/n)^n \leq e^{-t}$ where $n \geq t$, as claimed. □

Theorem VII.7.15

Theorem VII.7.15. If $\text{Re}(z) > 0$ then $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$.

Proof. Fix $x > 1$ and let $\varepsilon > 0$. By Lemma VII.7.16(b) there is $\kappa > 0$ such that

$$\int_\kappa^r e^{-t} t^{x-1}\, dt < \frac{\varepsilon}{4} \qquad (7.21)$$

whenever $r > \kappa$.

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$$\int_\kappa^r e^{-t} t^{x-1}\, dt < \frac{\varepsilon}{4} \tag{7.21}$$

whenever $r > \kappa$. Let $n \in \mathbb{N}$ satisfy $n > \kappa$ and let $f_n(x) = \int_{1/n}^n e^{-t} t^{z-1}\, dt$. Then

$$f_n(x) - \int_0^n e^{-t} t^{x-1}\, dt = -\int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt + \int_{1/n}^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)$$

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$$\int_\kappa^r e^{-t} t^{x-1}\, dt < \frac{\varepsilon}{4} \qquad (7.21)$$

whenever $r > \kappa$. Let $n \in \mathbb{N}$ satisfy $n > \kappa$ and let $f_n(x) = \int_{1/n}^n e^{-t} t^{z-1}\, dt$. Then

$$f_n(x) - \int_0^n e^{-t} t^{x-1}\, dt = -\int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt + \int_{1/n}^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)$$

Now

$$\begin{aligned} \int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt &\leq \int_0^{1/n} e^{-t} t^{x-1}\, dt \text{ by Lemma VII.7.19(b)} \\ &< \frac{\varepsilon}{4} \text{ for } n \text{ sufficiently large by Lemma VII.7.16(a).} \end{aligned} \qquad (7.22)$$

Theorem VII.7.15

Theorem VII.7.15. If $\mathrm{Re}(z) > 0$ then $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$.

Proof. Fix $x > 1$ and let $\varepsilon > 0$. By Lemma VII.7.16(b) there is $\kappa > 0$ such that

$$\int_\kappa^r e^{-t} t^{x-1}\, dt < \frac{\varepsilon}{4} \qquad (7.21)$$

whenever $r > \kappa$. Let $n \in \mathbb{N}$ satisfy $n > \kappa$ and let $f_n(x) = \int_{1/n}^n e^{-t} t^{z-1}\, dt$. Then

$$f_n(x) - \int_0^n e^{-t} t^{x-1}\, dt = -\int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt + \int_{1/n}^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)$$

Now

$$\begin{aligned} \int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt &\leq \int_0^{1/n} e^{-t} t^{x-1}\, dt \text{ by Lemma VII.7.19(b)} \\ &< \frac{\varepsilon}{4} \text{ for } n \text{ sufficiently large by Lemma VII.7.16(a).} \end{aligned} \qquad (7.22)$$

Theorem VII.7.15 (continued 1)

Proof (continued). Also if n is sufficiently large, by Lemma VII.7.19(a) and Proposition VII.1.19(b), on compact set $[0,\kappa]$ for $M = \int_0^\kappa t^{x-1}\,dt > 0$ we have $\left|\left(1-\frac{t}{n}\right)^n - e^{-t}\right| \le \frac{\varepsilon}{4M}$ for $t \in [0,\kappa]$. Then

$$\left|\int_{1/n}^{\kappa}\left(e^{-t}-\left(1-\frac{t}{n}\right)^n\right)t^{x-1}\,dt\right| \le \int_{1/n}^{\kappa}\left|e^{-t}-\left(1-\frac{t}{n}\right)^n\right| t^{x-1}\,dt$$

$$\le \frac{\varepsilon}{4M}\int_{1/n}^{\kappa} t^{x-1}\,dt \le \frac{\varepsilon}{4M}\int_0^{\kappa} t^{x-1}\,dt \le \frac{\varepsilon}{4M}M = \frac{\varepsilon}{4}. \qquad (7.23)$$

Theorem VII.7.15 (continued 1)

Proof (continued). Also if n is sufficiently large, by Lemma VII.7.19(a) and Proposition VII.1.19(b), on compact set $[0, \kappa]$ for $M = \int_0^\kappa t^{x-1}\,dt > 0$ we have $\left|\left(1 - \frac{t}{n}\right)^n - e^{-t}\right| \le \frac{\varepsilon}{4M}$ for $t \in [0, \kappa]$. Then

$$\left|\int_{1/n}^{\kappa} \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \le \int_{1/n}^{\kappa} \left|e^{-t} - \left(1 - \frac{t}{n}\right)^n\right| t^{x-1}\,dt$$

$$\le \frac{\varepsilon}{4M} \int_{1/n}^{\kappa} t^{x-1}\,dt \le \frac{\varepsilon}{4M} \int_0^{\kappa} t^{x-1}\,dt \le \frac{\varepsilon}{4M} M = \frac{\varepsilon}{4}. \qquad (7.23)$$

Next,

$$\left|\int_{\kappa}^{n} \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \le \int_{\kappa}^{n} \left|e^{-t} - \left(1 - \frac{t}{n}\right)^n\right| t^{x-1}\,dt$$

$$\le \int_{\kappa}^{n} \left(e^{-t} + \left(1 - \frac{t}{n}\right)^n\right) t^{x-1}\,dt$$

by the Triangle Inequality

Theorem VII.7.15 (continued 1)

Proof (continued). Also if n is sufficiently large, by Lemma VII.7.19(a) and Proposition VII.1.19(b), on compact set $[0,\kappa]$ for $M = \int_0^\kappa t^{x-1}\,dt > 0$ we have $\left|\left(1-\frac{t}{n}\right)^n - e^{-t}\right| \le \frac{\varepsilon}{4M}$ for $t \in [0,\kappa]$. Then

$$\left|\int_{1/n}^{\kappa}\left(e^{-t}-\left(1-\frac{t}{n}\right)^n\right)t^{x-1}\,dt\right| \le \int_{1/n}^{\kappa}\left|e^{-t}-\left(1-\frac{t}{n}\right)^n\right|t^{x-1}\,dt$$

$$\le \frac{\varepsilon}{4M}\int_{1/n}^{\kappa}t^{x-1}\,dt \le \frac{\varepsilon}{4M}\int_0^{\kappa}t^{x-1}\,dt \le \frac{\varepsilon}{4M}M = \frac{\varepsilon}{4}. \qquad (7.23)$$

Next,

$$\begin{aligned}\left|\int_{\kappa}^{n}\left(e^{-t}-\left(1-\frac{t}{n}\right)^n\right)t^{x-1}\,dt\right| &\le \int_{\kappa}^{n}\left|e^{-t}-\left(1-\frac{t}{n}\right)^n\right|t^{x-1}\,dt \\ &\le \int_{\kappa}^{n}\left(e^{-t}+\left(1-\frac{t}{n}\right)^n\right)t^{x-1}\,dt\end{aligned}$$

by the Triangle Inequality

Theorem VII.7.15 (continued 2)

Proof (continued).

$$\begin{aligned}\left|\int_\kappa^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| &\le \int_\kappa^n \left(e^{-t} + \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt \\ &= \int_\kappa^n 2e^{-t}t^{x-1}\,dt \\ &\qquad \text{by Lemma VII.7.19(b)} \\ &\le 2(\varepsilon/4) = \varepsilon/2 \text{ by (7.21)} \qquad (7.24)\end{aligned}$$

for $n > \kappa$. So

$$\begin{aligned}\left|f_n(x) - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right| &= \left|\int_{1/n}^n e^{-t}t^{x-1}\,dt \right. \\ &\quad \left. - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right| \\ &\quad \text{by the definition of } f_n\end{aligned}$$

Theorem VII.7.15 (continued 2)

Proof (continued).

$$\begin{aligned}\left|\int_\kappa^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| &\le \int_\kappa^n \left(e^{-t} + \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt \\ &= \int_\kappa^n 2e^{-t}t^{x-1}\,dt \\ &\qquad \text{by Lemma VII.7.19(b)} \\ &\le 2(\varepsilon/4) = \varepsilon/2 \text{ by (7.21)} \qquad (7.24)\end{aligned}$$

for $n > \kappa$. So

$$\begin{aligned}\left|f_n(x) - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right| &= \left|\int_{1/n}^n e^{-t}t^{x-1}\,dt\right. \\ &\qquad \left. - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right| \\ &\qquad \text{by the definition of } f_n\end{aligned}$$

Theorem VII.7.15 (continued 3)

Proof (continued).

$$\begin{aligned}
&\leq \int_0^{1/n} \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt + \left|\int_{1/n}^{n} \left(e^{-t}t^{x-1} - \left(1-\frac{t}{n}\right)^n t^{x-1}\right)\,dt\right| \\
&< \int_0^{1/n} \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt + \left|\int_{1/n}^{\kappa} \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \\
&\quad + \left|\int_{\kappa}^{n} \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \text{ by (7.22), (7.23), and (7.24)} \qquad (*)
\end{aligned}$$

for n sufficiently large. By Exercise VII.7.A, integration by parts yields

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = \frac{n!\,n^x}{x(x+1)\cdots(x+n)}.$$

Theorem VII.7.15 (continued 3)

Proof (continued).

$$\begin{aligned}
&\le \int_0^{1/n} \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt + \left|\int_{1/n}^{n} \left(e^{-t}t^{x-1} - \left(1-\frac{t}{n}\right)^n t^{x-1}\right)\,dt\right| \\
&< \int_0^{1/n} \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt + \left|\int_{1/n}^{\kappa} \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \\
&\quad + \left|\int_{\kappa}^{n} \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{x-1}\,dt\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \text{ by (7.22), (7.23), and (7.24)} \qquad (*)
\end{aligned}$$

for n sufficiently large. By Exercise VII.7.A, integration by parts yields

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Theorem VII.7.15 (continued 4)

Proof (continued). Combining this with (∗) gives

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(f_n(x) - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right) \\ &= \lim_{n\to\infty}\left(f_n(x) - \frac{n!n^x}{x(x+1)\cdots(x+n)}\right) \\ &= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)} \\ &= f(x) - \Gamma(x) \text{ by Gauss's Formula (Theorem VII.7.6).} \end{aligned}$$

So $f(x) = \Gamma(x)$ for $x > 1$. Since f and γ are both analytic on $G = \{z \mid \mathrm{Re}(z) > 0\}$ and $\{x \mid x > 1\} \subset \mathbb{R}$ has a limit point in G, then by Corollary IV.3.8, $f = \Gamma$ on G, as claimed. □

Theorem VII.7.15 (continued 4)

Proof (continued). Combining this with $(*)$ gives

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(f_n(x) - \int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt\right) \\ &= \lim_{n\to\infty}\left(f_n(x) - \frac{n!n^x}{x(x+1)\cdots(x+n)}\right) \\ &= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)} \\ &= f(x) - \Gamma(x) \text{ by Gauss's Formula (Theorem VII.7.6).} \end{aligned}$$

So $f(x) = \Gamma(x)$ for $x > 1$. Since f and γ are both analytic on $G = \{z \mid \mathrm{Re}(z) > 0\}$ and $\{x \mid x > 1\} \subset \mathbb{R}$ has a limit point in G, then by Corollary IV.3.8, $f = \Gamma$ on G, as claimed. □