

## Lemma VII.8.3

## Lemma VII.8.3.

- (a) Let  $S = \{z \mid \operatorname{Re}(z) \geq a\}$  where  $a > 1$ . If  $\varepsilon > 0$  then there is a number  $\delta$ ,  $0 < \delta < 1$ , such that for all  $z \in S$  we have

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon$$

whenever  $\delta > \beta > \alpha > 0$ .

- (b) Let  $S = \{z \mid \operatorname{Re}(z) \leq A\}$  where  $A \in \mathbb{R}$ . If  $\varepsilon > 0$  then there is a number  $\kappa > 1$  such that for all  $z \in S$  we have

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon$$

whenever  $\beta > \alpha > \kappa$ .

## Complex Analysis

## Chapter VII. Compactness and Convergence in the Space of Analytic Functions

## VII.8. The Riemann Zeta Function—Proofs of Theorems



## Lemma VII.8.3 (continued 1)

**Proof.** (a) Since  $e^t - 1 \geq t$  for all  $t \geq 0$ , we have that for  $0 < t \leq 1$  and for  $z \in S$

$$\begin{aligned} |(e^t - 1)^{-1} t^{z-1}| &\leq t |t^{z-1}| \\ &= t^{-1} t^{\operatorname{Re}(z)-1} \text{ see the second note of this section: } \\ &\quad n^z = z^{\operatorname{Re}(z)} \\ &\leq t^{-1} t^{a-1} \text{ since } \operatorname{Re}(z) \geq a \text{ and } 0 < t \leq 1 \\ &= t^{a-2}. \end{aligned}$$

Since  $a > 1$ ,  $\int_0^1 t^{a-2} dz = \frac{1}{a-1} t^{a-1} \Big|_0^1 = \frac{1}{a-1} < \infty$ . Let  $\varepsilon > 0$ . There

exists  $\delta > 0$  such that for  $\delta > \beta > \alpha > 0$  we have  $\left| \int_{\alpha}^{\beta} t^{a-2} dt \right| < \varepsilon$  (if not, then  $\int_0^1 t^{a-2} dt$  can be made arbitrarily large). So

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| \leq \left| \int_{\alpha}^{\beta} t^{a-2} dt \right| < \varepsilon.$$

## Lemma VII.8.3 (continued 2)

**Proof.** (b) If  $t > 1$  and  $z \in S = \{z \mid \operatorname{Re}(z) \leq A\}$ ,  $A \in \mathbb{R}$ , we have

$$\begin{aligned} |t^{z-1}| &\leq t^{\operatorname{Re}(z)-1} \text{ as in (a)} \\ &= \leq t^{A-1} \text{ since } \operatorname{Re}(z) \leq A. \end{aligned}$$

Since  $t^{A-1} \exp(-t/2)$  is continuous for  $t \in [1, \infty)$  and converges to 0 as  $t \rightarrow \infty$ , there is a constant  $c$  such that  $t^{A-1} \exp(-t/2) \leq c$  for all  $t \geq 1$ . If you like,  $t^{A-1} \exp(-t/2)$  has a maximum of  $c$  for  $t \in [1, \infty)$ :  $t^{A-1} \exp(-t/2) \leq c$  or  $T^{A-1} \leq c \exp(t/2)$ . So

$$|(e^t - 1)^{-1} t^{z-1}| \leq (e^t - 1)^{-1} t^{A-1} \leq (e^t - 1)^{-1} c \exp(t/2).$$

Since  $e^{t/2}(e^t - 1)^{-1}$  is integrable over  $t \in [1, \infty)$  (use  $u$  substitution), then  $\kappa > 1$  can be found as required (otherwise the integral can be made arbitrarily large).  $\square$

## Corollary VII.8.4

Corollary VII.8.4.

- (a) If  $S = \{z \mid a \leq \operatorname{Re}(z) \leq A\}$  where  $1 < a < A < \infty$  then the integral

$$\int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on  $S$ .

- (b) If  $S = \{z \mid \operatorname{Re}(z) \leq A\}$  where  $-\infty < A < \infty$ , then the integral

$$\int_1^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on  $S$ .

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## Corollary VII.8.4 (continued 2)

Corollary VII.8.4.

- (a) If  $S = \{z \mid a \leq \operatorname{Re}(z) \leq A\}$  where  $1 < a < A < \infty$  then the integral

$$\int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on  $S$ .

**Proof of (a).** The proof here is similar to the proof of (b). First, find an  $M$  as in the proof of (b) (where we need  $\operatorname{Re}(z) \leq A$ ). Second, find a  $\delta > 0$  as guaranteed by Lemma VII.8.3(a) (where we need  $a \leq \operatorname{Re}(z)$ ).  $\square$

## Corollary VII.8.4 (continued 1)

**Proof of (b).** Let  $\varepsilon > 0$  be given. We need to show that there is  $M \geq 1$  such that for all  $N \geq M$  we have

$$\begin{aligned} & \left| \int_1^{\infty} (e^t - 1)^{-1} t^{z-1} dt - \int_1^N (e^t - 1)^{-1} t^{z-1} dt \right| \\ &= \left| \int_N^{\infty} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon. \end{aligned}$$

By Lemma VII.8.3(a), there is  $\kappa > 1$  with

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon/2$$

for all  $\beta > \alpha > \kappa$ . So take  $M = \kappa + 1$  so that for  $N \geq M$  we have

$$\left| \int_N^{\infty} (e^t - 1)^{-1} t^{z-1} dt \right| \leq \varepsilon/2 < \varepsilon$$

and (b) follows (for  $\operatorname{Re}(z) \leq A$ , as required).

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## Proposition VII.8.5

**Proposition VII.8.5.** For  $\operatorname{Re}(z) > 1$

$$\zeta(z)\Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt.$$

**Proof.** By the note above,  $\int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > 1\}$ . So  $\zeta(z)\Gamma(z)$  and the integral are both analytic functions on  $\{z \mid \operatorname{Re}(z) > 1\}$ . By Corollary IV.3.8, if we can show these functions agree on a set with a limit point in  $\{z \mid \operatorname{Re}(z) > 1\}$  then the equality is established. We so so for  $z \in \{x \in \mathbb{R} \mid x > 1\}$ .

By Lemma VII.8.3, there are  $\alpha$  and  $\beta$  where  $0 < \alpha < \beta < \infty$  such that:

$$\int_0^{\alpha} (e^t - 1)^{-1} t^{x-1} dt < \frac{\varepsilon}{4} \quad \text{and} \quad \int_{\beta}^{\infty} (e^t - 1)^{-1} t^{x-1} dt < \frac{\varepsilon}{4}.$$

For all  $n \in \mathbb{N}$  we have

$$\sum_{k=1}^n e^{-kt} \leq \sum_{k=1}^{\infty} e^{-kt} = \sum_{k=1}^{\infty} \left(\frac{1}{e^t}\right)^k = \frac{1/e^t}{1 - 1/e^t} = \frac{1}{e^t - 1} = (e^t - 1)^{-1}.$$

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## Proposition VII.8.5 (continued 1)

**Proof (continued).** So

$$\begin{aligned} \sum_{k=1}^n \left( \int_0^\alpha e^{-kt} t^{x-1} dt \right) &= \int_0^\alpha \left( \sum_{k=1}^n e^{-kt} t^{x-1} \right) dt \\ &\leq \int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt \leq \frac{\varepsilon}{4} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^n \left( \int_\beta^\infty e^{-kt} t^{x-1} dt \right) &= \int_\beta^\infty \left( \sum_{k=1}^n e^{-kt} t^{x-1} \right) dt \\ &\leq \int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{\varepsilon}{4} \end{aligned}$$

for all  $a \in \mathbb{N}$ . Letting  $n \rightarrow \infty$  (and replacing index  $k$  with  $n$ ) we get...

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Proposition VII.8.5

## Proposition VII.8.5 (continued 3)

**Proof (continued).**

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left| \sum_{n=1}^\infty \left( \int_\alpha^\beta e^{-nt} t^{x-1} dt \right) - \int_\alpha^\beta (e^t - 1)^{-1} t^{x-1} dt \right| \quad (*)$$

(all integrals are positive, so we have absolute convergence of the series and can rearrange). Now  $\sum_{n=1}^\infty e^{-nt}$  converges uniformly on  $[\alpha, \beta]$  (pointwise convergence on a compact set) so

$$\sum_{n=1}^\infty \left( \int_\alpha^\beta e^{-nt} t^{x-1} dt \right) = \int_\alpha^\beta \left( \sum_{n=1}^\infty e^{-nt} \right) t^{x-1} dt = \int_\alpha^\beta (e^t - 1)^{-1} t^{x-1} dt$$

and (\*) implies the result.  $\square$

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## Proposition VII.8.5 (continued 2)

**Proof (continued).**

$$\sum_{n=1}^\infty \left( \int_0^\infty e^{-nt} t^{x-1} dt \right) < \frac{\varepsilon}{4} \text{ and } \sum_{n=1}^\infty \left( \int_\beta^\infty e^{-nt} t^{x-1} dt \right) < \frac{\varepsilon}{4}.$$

Equation (8.2) gives  $\zeta(x)\Gamma(x) = \sum_{n=1}^\infty \left( \int_0^\infty e^{-nt} t^{x-1} dx \right)$  so

$$\begin{aligned} &\left| \zeta(z)\Gamma(x) - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt \right| \\ &= \left| \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{x-1} dt - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt \right| \\ &= \left| \sum_{n=1}^\infty \left( \int_0^\alpha e^{-nt} t^{x-1} dt + \int_\alpha^\beta e^{-nt} t^{x-1} dt + \int_\beta^\infty e^{-nt} t^{x-1} dt \right) \right. \\ &\quad \left. - \int_\alpha^\beta (e^t - 1)^{-1} t^{x-1} dt - \int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt \right| \end{aligned}$$

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Lemma VII.8.C

## Theorem VII.8.C

**Lemma VII.8.C.** The function  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > 0\}$ .

**Proof.** Since  $\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^\infty a_n z^n$  for some  $a_1, a_2, \dots$  then

$\frac{1}{e^t - 1} - \frac{1}{t}$  has a limit of  $-1/2$  as  $t \rightarrow 0$ . So  $\frac{1}{e^t - 1} - \frac{1}{t}$  is bounded in a deleted neighborhood of  $t = 0$  and so is bounded on  $(0, 1]$ , say by  $M$ . Let  $K$  be a compact subset of  $\{z \mid \operatorname{Re}(z) > 0\}$  and let  $m = \min\{\operatorname{Re}(z) \mid z \in K\}$ . For  $z \in K$ , consider

$$\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt = \lim_{a \rightarrow 0^+} \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt.$$

Let  $\varepsilon > 0$  and choose  $\delta = \min\{(n\varepsilon/M)^{1/m}, 1\}$ .

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## Theorem VII.8.C (continued 1)

**Proof (continued).** Then for  $0 < a < \delta < 1$  we have

$$\begin{aligned} & \left| \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt - \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \right| \\ &= \left| \int_0^a \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \right| \leq \int_0^1 \left| \frac{1}{e^t - 1} - \frac{1}{t} \right| |t^{z-1}| dt \\ &\leq M \int_0^a |e^{(z-1)\log t}| dt = M \int_0^a e^{(\operatorname{Re}(z)-1)\log t} dt \\ &= M \int_0^a t^{\operatorname{Re}(z)-1} dt = M \frac{1}{\operatorname{Re}(z)} t^{\operatorname{Re}(z)} \Big|_0^a \\ &= \frac{M}{\operatorname{Re}(z)} a^{\operatorname{Re}(z)} < \frac{M}{\operatorname{Re}(z)} \delta^{\operatorname{Re}(z)} < \frac{M}{m} \delta^m \\ &= \frac{M}{m} \left( \left( \frac{m\varepsilon}{M} \right)^{1/m} \right)^m = \frac{M m \varepsilon}{m} = \varepsilon. \end{aligned}$$

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Lemma VII.8.D

## Theorem VII.8.D

**Lemma VII.8.D.** The function  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > -1\}$ .

**Proof.** Let  $K$  be a compact subset of  $G = \{z \mid \operatorname{Re}(z) > -1\}$  and let  $m = \min\{\operatorname{Re}(z) \mid z \in K\}$ . For  $z \in K$  consider

$$\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt = \lim_{a \rightarrow 0^+} \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt.$$

Let  $\varepsilon > 0$  and choose  $\delta = \min\{(\varepsilon(m+1))^{1/(m+1)}, 1\}$ . Then for  $0 < a < \delta < 1$  we have

$$\left| \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt - \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt \right|$$

## Theorem VII.8.C (continued 2)

**Lemma VII.8.C.** The function  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > 0\}$ .

**Proof (continued).** So, since  $\delta$  is independent of  $z \in K$ , then

$$\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt = \lim_{a \rightarrow 0^+} \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$$

converges uniformly on  $K$ . Since  $K$  is an arbitrary compact subset of  $G$ , then  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$  is uniformly convergent on compact subsets of  $G$ . So by Corollary VII.8.B,  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$  is an analytic function on  $\{z \mid \operatorname{Re}(z) > 0\}$ .  $\square$

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Lemma VII.8.D

## Theorem VII.8.D (continued 1)

**Proof (continued).**

$$\begin{aligned} &= \left| \int_0^a \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt \right| \leq \int_0^1 \left| \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right| |t^{z-1}| dt \\ &\leq \int_0^a |t| |t^{z-1}| dt \text{ since } \frac{1}{e^t - 1} + \frac{1}{2} \leq ct \text{ for some constant } c \text{ and} \end{aligned}$$

for  $t \in [0, 1]$  by Exercise VII.8.A

$$\begin{aligned} &= c \int_0^a |t^z| dt - \int_0^a |e^{z \log t}| dt = \int_0^a e^{\operatorname{Re}(z) \log t} dt \\ &= \int_0^a t^{\operatorname{Re}(z)} dt = \frac{1}{\operatorname{Re}(z)+1} t^{\operatorname{Re}(z)+1} \Big|_{t=0}^{t=a} \\ &= \frac{\operatorname{Re}(z)+1}{a^{\operatorname{Re}(z)+1}} \leq \frac{m+1}{m+1} < \frac{\delta^{\operatorname{Re}(z)+1}}{\delta^{m+1}} < \frac{m+1}{m+1} \\ &\leq \frac{(\varepsilon(m+1))^{1/(m+1)} m+1}{m+1} = \frac{\varepsilon(m+1)}{m+1} = \varepsilon. \end{aligned}$$

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## Theorem VII.8.D (continued 2)

**Lemma VII.8.D.** The function  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > -1\}$ .

**Proof (continued).** So, since  $\delta$  is independent of  $z \in K$ , then

$$\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt = \lim_{a \rightarrow 0^+} \int_a^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$$

converges uniformly on  $K$ . Since  $K$  is an arbitrary compact subset of  $G$ , then  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$  is uniformly convergent on compact subsets of  $G$ . So by Corollary VII.8.B,  $\int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) > -1\}$ .  $\square$

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## Theorem VII.8.E

**Lemma VII.8.E.** The function  $\int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) < 1\}$ .

**Proof.** We have  $\lim_{t \rightarrow \infty} t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) = 1$  by L'hospital's Rule, so

$t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right)$  is bounded for  $t$  sufficiently large, say  $t \geq N$ . Also,  $t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right)$  is continuous from  $t > 0$ , so it is bounded on  $[1, \infty)$ , say  $t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) \leq M$  for  $t \geq 1$ . Let  $K$  be a compact subset of  $G = \{z \mid \operatorname{Re}(z) < 1\}$  and let  $M' = \max\{\operatorname{Re}(z) \mid z \in K\}$ . For  $z \in K$  consider

$$\int_1^\infty t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt = \lim_{b \rightarrow \infty} \left( \int_1^b t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt \right).$$

Let  $\varepsilon > 0$ . Choose  $N' = \max \left\{ N, \left( \frac{(1-M')\varepsilon}{M} \right)^{1/(\operatorname{Re}(z)-2)} \right\}$ .

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## Theorem VII.8.E (continued 1)

**Proof (continued).** Then for  $b > N'$  we have

$$\left| \int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt - \int_1^b \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \right|$$

$$= \left| \int_b^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \right| \leq \int_b^\infty \left| \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t \right| |t^{z-2}| dt$$

$$\leq M \int_b^\infty \left| e^{(\operatorname{Re}(z)-2) \log t} \right| dt = M \int_b^\infty e^{(\operatorname{Re}(z)-2) \log t} dt$$

$$\leq M \int_b^\infty t^{\operatorname{Re}(z)-2} dt = M \frac{1}{\operatorname{Re}(z)-1} \left| t^{\operatorname{Re}(z)-1} \right|_b^\infty$$

$$= M \frac{1}{\operatorname{Re}(z)-1} \left( \lim_{c \rightarrow \infty} c^{\operatorname{Re}(z)-1} \right) - b^{\operatorname{Re}(z)-1}$$

$$= M \frac{1}{\operatorname{Re}(z)-1} b^{\operatorname{Re}(z)-1} < \frac{M}{1 - \operatorname{Re}(z)} (N')^{\operatorname{Re}(z)-1}$$

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## Theorem VII.8.E (continued 2)

**Proof (continued).**

$$\leq \frac{M}{1 - M'} (N')^{\operatorname{Re}(z)-1} \text{ since } \operatorname{Re}(z) \leq M' \text{ implies } \frac{1}{1 - \operatorname{Re}(z)} = \varepsilon$$

$$\leq \frac{M}{1 - M'} \left( \left( \frac{(1 - M')\varepsilon}{M} \right)^{1/(\operatorname{Re}(z)-1)} \right)^{\operatorname{Re}(z)-1} - \frac{M}{1 - M'} \frac{(1 - M')\varepsilon}{M} = \varepsilon.$$

So, since  $N'$  is independent of  $z \in K$ , then

$$\int_1^\infty t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt = \lim_{b \rightarrow \infty} \left( \int_1^b t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt \right)$$

converges uniformly on  $K$ . Since  $K$  is an arbitrary compact subset of  $G$ , then  $\int_1^\infty t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt$  is uniformly convergent on compact subsets of  $G$ . So by Theorem VII.8.A/Exercise VIII.2.2  $\int_1^\infty t \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) t^{z-1} dt$  is an analytic function on  $G = \{z \mid \operatorname{Re}(z) < 1\}$ .  $\square$

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## Theorem VII.8.13

**Theorem VII.8.13. Riemann's Functional Equation.**

For  $-1 < \operatorname{Re}(z) < 0$  we have

$$\zeta(z) = 2(2\pi)^{z-1} \Gamma(1-z) \zeta(1-z) \sin \pi z / 2.$$

**Proof.** If  $-1 < \operatorname{Re}(z) < 0$  then

$$\int_1^\infty t^{z-1} dt = \frac{1}{z} t^z \Big|_1^\infty = \lim_{b \rightarrow \infty} \frac{1}{z} b^z - \frac{1}{z} = 0 - \frac{1}{z} = -\frac{1}{z}$$

(notice  $\lim_{b \rightarrow \infty} |b^z| = \lim_{b \rightarrow \infty} b^{\operatorname{Re}(z)} = 0$  since  $-1 < \operatorname{Re}(z) < 0$ ). So for  $-1 < \operatorname{Re}(z) < 0$ , we can write (8.8) as

$$\begin{aligned} \zeta(z) \Gamma(z) &= \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt + \frac{1}{2} \int_1^\infty t^{z-1} dt \\ &\quad + \int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \end{aligned}$$

## Theorem VII.8.13 (continued 2)

**Proof (continued).** By Exercise V.2.8, for  $a \notin \mathbb{Z}$  we have

$$\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^\infty \frac{2a}{a^2 - n^2} \quad \text{so that with } a = it/(2\pi) \text{ we have}$$

$$\pi \cot(it/2) = \frac{2\pi}{it} + \sum_{n=1}^\infty \frac{it/\pi}{\frac{-t^2}{4\pi^2} - n^2} \quad \text{or}$$

$$\cot(it/2) = \frac{2}{it} + \sum_{n=1}^\infty \frac{it}{it^2/4 - \pi^2 n^2} = \frac{2}{it} - 4it \sum_{n=1}^\infty \frac{1}{t^2 + 4\pi^2 n^2}$$

where  $it/(2\pi) \notin \mathbb{Z}$  (we'll have  $t$  real, so we need only avoid  $t = 0$ ). Then

$$\begin{aligned} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{1}{t} &= \left( \frac{i}{2} \cot(it/2) - \frac{1}{t} \right) \frac{1}{t} \\ &= \left( \frac{i}{2} \left( \frac{2}{it} - 4it \sum_{n=1}^\infty \frac{1}{t^2 + 4\pi^2 n^2} \right) - \frac{1}{t} \right) \frac{1}{t} \end{aligned}$$

## Theorem VII.8.13 (continued 1)

**Proof (continued).** . . . so

$$\zeta(z) \Gamma(z) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt \quad \text{for } -1 < \operatorname{Re}(z) < 0. \quad (8.9)$$

Notice that (see page 32)

$$\cot z = \frac{\cos z}{\sin z} = \frac{(e^{-iz} + e^{-iz})/2}{2(e^{iz} - e^{-iz})/(2i)} = i \frac{1 + e^{-2iz}}{1 - e^{-2iz}}$$

and with  $z = -t/(2i) = it/2$  we have  $\cot\left(\frac{-1}{2i}t\right) = \cot(it/2) = i \frac{1 + e^t}{1 - e^t}$

or  $\frac{i}{2} \cot(it/2) = -\frac{1 + e^t}{2(1 - e^t)} = \frac{1 + e^t}{2e^t - 1}$ . So we have

$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{2 + e^t - 1}{2(e^t - 1)} = \frac{e^t + 1}{2(e^t - 1)} = \frac{i}{2} \cot(it/2).$$

## Theorem VII.8.13 (continued 3)

**Proof (continued).**

$$= \left( \frac{1}{t} + 2t \sum_{n=1}^\infty \frac{1}{t^2 + 4\pi^2 n^2} - \frac{1}{t} \right) \frac{1}{t} = 2 \sum_{n=1}^\infty \frac{1}{t^2 + 4\pi^2 n^2},$$

so that (8.9) becomes, for  $-1 < \operatorname{Re}(z) < 0$ ,

$$\begin{aligned} \zeta(z) \Gamma(z) &= \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt - \frac{1}{2z} + \int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \\ &= \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt + \frac{1}{2} \int_1^\infty t^{z-1} dt \\ &\quad + \int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt \end{aligned}$$

$$= \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt = 2 \int_0^\infty \left( \sum_{n=1}^\infty \frac{1}{t^2 + 4\pi^2 n^2} \right) t^z dt.$$

## Theorem VII.8.13 (continued 4)

**Proof (continued).** We leave "it to the reader to justify the interchanging of the sum and the integral" (Conway says on page 192; so one must show that the series converges uniformly for  $t \in [0, \infty)$ ). We then have for  $-1 < \operatorname{Re}(z) < 0$

$$\zeta(z)\Gamma(z) = 2 \sum_{n=1}^{\infty} \left( \int_0^{\infty} \frac{t^z}{t^2 + 4\pi^2 n^2} dt \right) = 2 \sum_{n=1}^{\infty} \left( \int_0^{\infty} \frac{t^z}{t^2 + (2\pi n)^2} dt \right)$$

Let  $t = 2\pi n t'$  so that  $t' \in [0, \infty)$  and  $dt = 2\pi n dt'$

$$\begin{aligned} &= 2 \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(2\pi n t')^z}{(2\pi n t')^2 + 2\pi n)^2} 2\pi n dt' \\ &= 2 \sum_{n=1}^{\infty} (2\pi n)^{z-1} \left( \int_0^{\infty} \frac{(t')^z}{(t')^2 + 1} dt' \right) \\ &= 2 \sum_{n=1}^{\infty} (2\pi n)^{z-1} \left( \int_0^{\infty} \frac{t^z}{t^2 + 1} dt \right) \end{aligned}$$

## Theorem VII.8.13 (continued 5)

**Proof (continued).**

$$\begin{aligned} \zeta(x) &= \frac{1}{\Gamma(x)} 2(2\pi)^{x-1} \zeta(1-x) \int_0^{\infty} \frac{t^x}{t^2 + 1} dt \\ &= \frac{1}{\Gamma(x)} 2(2\pi)^{x-1} \zeta(1-x) \frac{1}{2} \pi \sec(\pi x/2) \\ &= \left( \frac{\Gamma(1-x)}{\pi} 2 \sin(\pi x/2) \cos(\pi x/2) \right) \left( 2(2\pi)^{x-1} \zeta(1-x) \frac{1}{2} \pi \sec(\pi x/2) \right) \\ &= 2(2\pi)^{x-1} \Gamma(1-x) \zeta(1-x) \sin(\pi x/2) \text{ for } -1 < x < 0. \end{aligned}$$

So we have defined  $\zeta(z)$  and  $2(2\pi)^{z-1} \Gamma(1-z) \zeta(1-z) \sin(\pi z/2)$  as analytic functions in  $-1 < \operatorname{Re}(z) < 0$ . Since they are equal on  $-1 < x < 0$  then, by Corollary IV.3.8, they are equal on  $-1 < \operatorname{Re}(z) < 0$ .  $\square$

## Theorem VII.8.13 (continued 4)

**Proof (continued).**

$$\begin{aligned} \zeta(z)\Gamma(z) &= 2(2\pi)^{z-2} \left( \int_0^{\infty} \frac{t^z}{t^2 + 1} dt \right) \sum_{n=1}^{\infty} n^{z-1} \\ &= 2(2\pi)^{z-1} \zeta(1-z) \int_0^{\infty} \frac{t^z}{t^2 + 1} dt \end{aligned} \quad (8.11)$$

since  $\operatorname{Re}(1-z) > 1$  and by

definition  $\zeta(1-z) = \sum_{n=1}^{\infty} n^{-(1-z)} = \sum_{n=1}^{\infty} n^{z-1}$ . For  $z \in \mathbb{R}$  where  $-1 < x < 0$ , by Example V.2.12 we have

$$\int_0^{\infty} \frac{t^x}{t^2 + 1} dt = \frac{1}{2} \int_0^{\infty} \frac{s^{(z-1)/2}}{s+1} ds = \frac{\pi}{2} \cos \frac{\pi(1-x)}{2} = \frac{\pi}{2} \sec \frac{\pi x}{2}. \quad (8.12)$$

By Exercise VII.7.2,

$$\frac{1}{\Gamma(x)} = \frac{\Gamma(1-x)}{\pi} \sin(\pi x) - \frac{\Gamma(1-x)}{\pi} 2 \sin(\pi x/2) \cos(\pi x/2),$$

so from (8.11) we have (for  $z = x$ )...

## Theorem VII.8.17

**Theorem VII.8.17. Euler's Theorem.**

If  $\operatorname{Re}(z) > 1$  then

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$

where  $\{p_n\}$  is the sequence of prime numbers.

**Proof.** For each  $n \in \mathbb{N}$  we have  $\sum_{m=0}^{\infty} p_n^{-mz} = \frac{1}{1 - p_n^{-z}}$  since this is a geometric series with ration  $p_n^{-z}$  (and  $|p_n^{-z}| = p_n^{\operatorname{Re}(-z)} < 1$ ). For each  $n \in \mathbb{N}$ ,

$$\prod_{k=1}^n \frac{1}{1 - p_k^{-z}} = \prod_{k=1}^n \left( \sum_{m=0}^{\infty} p_n^{-mz} \right).$$

Now the  $n$  series on the right hand side converge absolutely (they are geometric series) and so can be rearranged. So the right hand side includes all elements of  $\mathbb{N}$  to the  $z$  power which are products of powers of the primes  $p_1, p_2, \dots, p_n$ .

## Theorem VII.8.17 (continued)

**Proof (continued).** Denote these natural numbers as  $n_1, n_2, \dots$ . Notice that each such  $n_j$  appears only once by the Fundamental Theorem of Arithmetic. So

$$\prod_{k=1}^n \frac{1}{1 - p_k^{-z}} = \sum_{j=1}^{\infty} n_j^{-z}.$$

So letting  $n \rightarrow \infty$  then all primes are included and so all natural numbers are included so that

$$\prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-z}} = \prod_{k=1}^{\infty} \left( \sum_{m=0}^{\infty} p_k^{-mz} \right) = \sum_{j=1}^{\infty} n_j^{-z} = \zeta(z).$$

□