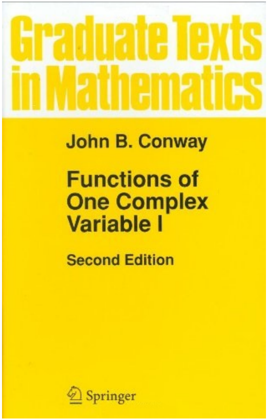
Complex Analysis

Chapter VII. Compactness and Convergence in the Space of Analytic Functions

VII.8. The Riemann Zeta Function—Proofs of Theorems



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Lemma VII.8.3

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(a) Let $S = \{z \mid \mathrm{Re}(z) \geq a\}$ where $a > 1$. If $\varepsilon > 0$ then there is a number δ, $0 < \delta < 1$, such that for all $z \in S$ we have

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\, dt \right| < \varepsilon$$

whenever $\delta > \beta > \alpha > 0$.

(b) Let $S = \{z \mid \mathrm{Re}(z) \leq A\}$ where $A \in \mathbb{R}$. If $\varepsilon > 0$ then there is a number $\kappa > 1$ such that for all $z \in S$ we have

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\, dt \right| < \varepsilon$$

whenever $\beta > \alpha > \kappa$.

Lemma VII.8.3 (continued 1)

Proof. (a) Since $e^t - 1 \geq t$ for all $t \geq 0$, we have that for $0 < t \leq 1$ and for $z \in S$

$$\begin{aligned} |(e^t - 1)^{-1} t^{z-1}| &\leq t|t^{z-1}| \\ &= t^{-1} t^{\text{Re}(z)-1} \text{ see the second note of this section: } n^z = z^{\text{Re}(z)} \\ &\leq t^{-1} t^{a-1} \text{ since } \text{Re}(z) \geq a \text{ and } 0 < t \leq 1 \\ &= t^{a-2}. \end{aligned}$$

Since $a > 1$, $\int_0^1 t^{a-2}\, dz = \frac{1}{a-1} t^{a-1}\Big|_0^1 = \frac{1}{a-1} < \infty$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $\delta > \beta > \alpha > 0$ we have $\left|\int_\alpha^\beta t^{a-2}\, dt\right| < \varepsilon$ (if not, then $\int_0^1 t^{a-2}\, dt$ can be made arbitrarily large).

Lemma VII.8.3 (continued 1)

Proof. (a) Since $e^t - 1 \geq t$ for all $t \geq 0$, we have that for $0 < t \leq 1$ and for $z \in S$

$$\begin{array}{rcl} |(e^t-1)^{-1}t^{z-1}| & \leq & t|t^{z-1}| \\ & = & t^{-1}t^{\text{Re}(z)-1} \text{ see the second note of this section: } n^z = z^{\text{Re}(z)} \\ & \leq & t^{-1}t^{a-1} \text{ since Re}(z) \geq a \text{ and } 0 < t \leq 1 \\ & = & t^{a-2}. \end{array}$$

Since $a > 1$, $\displaystyle\int_0^1 t^{a-2}\,dz = \frac{1}{a-1}t^{a-1}\Big|_0^1 = \frac{1}{a-1} < \infty$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $\delta > \beta > \alpha > 0$ we have $\left|\int_\alpha^\beta t^{a-2}\,dt\right| < \varepsilon$ (if not, then $\int_0^1 t^{a-2}\,dt$ can be made arbitrarily large). So

$$\left|\int_\alpha^\beta (e^t-1)^{-1}t^{z-1}\,dt\right| \leq \left|\int_\alpha^\beta t^{a-2}\,dt\right| < \varepsilon.$$

Lemma VII.8.3 (continued 1)

Proof. (a) Since $e^t - 1 \geq t$ for all $t \geq 0$, we have that for $0 < t \leq 1$ and for $z \in S$

$$\begin{aligned} |(e^t-1)^{-1}t^{z-1}| &\leq t|t^{z-1}| \\ &= t^{-1}t^{\text{Re}(z)-1} \text{ see the second note of this section: } \\ & \qquad\qquad n^z = z^{\text{Re}(z)} \\ &\leq t^{-1}t^{a-1} \text{ since } \text{Re}(z) \geq a \text{ and } 0 < t \leq 1 \\ &= t^{a-2}. \end{aligned}$$

Since $a > 1$, $\displaystyle\int_0^1 t^{a-2}\,dz = \frac{1}{a-1}t^{a-1}\Big|_0^1 = \frac{1}{a-1} < \infty$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $\delta > \beta > \alpha > 0$ we have $\left|\int_\alpha^\beta t^{a-2}\,dt\right| < \varepsilon$ (if not, then $\int_0^1 t^{a-2}\,dt$ can be made arbitrarily large). So

$$\left|\int_\alpha^\beta (e^t-1)^{-1}t^{z-1}\,dt\right| \leq \left|\int_\alpha^\beta t^{a-2}\,dt\right| < \varepsilon.$$

Lemma VII.8.3 (continued 2)

Proof. (b) If $t > 1$ and $z \in S = \{z \mid \text{Re}(z) \leq A\}$, $A \in \mathbb{R}$, we have

$$\begin{aligned} |t^{z-1}| &\leq t^{\text{Re}(z)-1} \text{ as in (a)} \\ &= \leq t^{A-1} \text{ since } \text{Re}(z) \leq A. \end{aligned}$$

Since $t^{A-1} \exp(-t/2)$ is continuous for $t \in [1, \infty)$ and converges to 0 as $t \to \infty$, there is a constant c such that $t^{A-1} \exp(-t/2) \leq c$ for all $t \geq 1$. If you like, $t^{A-1} \exp(-t/2)$ has a maximum of c for $t \in [1, \infty)$: $t^{A-1} \exp(-t/2) \leq c$ or $T^{A-1} \leq c \exp(t/2)$.

Lemma VII.8.3 (continued 2)

Proof. (b) If $t > 1$ and $z \in S = \{z \mid \text{Re}(z) \leq A\}$, $A \in \mathbb{R}$, we have

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$$|(e^t - 1)^{-1} t^{z-1}| \leq (e^t - 1)^{-1} t^{A-1} \leq (e^t - 1)^{-1} c \exp(t/2).$$

Since $e^{t/2}(e^t - 1)^{-1}$ is integrable over $t \in [1, \infty)$ (use u substitution), then $\kappa > 1$ can be found as required (otherwise the integral can be made arbitrarily large). □

Lemma VII.8.3 (continued 2)

Proof. (b) If $t > 1$ and $z \in S = \{z \mid \text{Re}(z) \le A\}$, $A \in \mathbb{R}$, we have

$$\begin{aligned} |t^{z-1}| &\le t^{\text{Re}(z)-1} \text{ as in (a)} \\ &= \le t^{A-1} \text{ since } \text{Re}(z) \le A. \end{aligned}$$

Since $t^{A-1}\exp(-t/2)$ is continuous for $t \in [1, \infty)$ and converges to 0 as $t \to \infty$, there is a constant c such that $t^{A-1}\exp(-t/2) \le c$ for all $t \ge 1$. If you like, $t^{A-1}\exp(-t/2)$ has a maximum of c for $t \in [1, \infty)$: $t^{A-1}\exp(-t/2) \le c$ or $T^{A-1} \le c\exp(t/2)$. So

$$|(e^t - 1)^{-1}t^{z-1}| \le (e^t - 1)^{-1}t^{A-1} \le (e^t - 1)^{-1}c\exp(t/2).$$

Since $e^{t/2}(e^t - 1)^{-1}$ is integrable over $t \in [1, \infty)$ (use u substitution), then $\kappa > 1$ can be found as required (otherwise the integral can be made arbitrarily large). □

Corollary VII.8.4

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(a) If $S = \{z \mid a \leq \mathrm{Re}(z) \leq A\}$ where $1 < a < A < \infty$ then the integral
$$\int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$$
converges uniformly on S.

(b) If $S = \{z \mid \mathrm{Re}(z) \leq A\}$ where $-\infty < A < \infty$, then the integral
$$\int_1^\infty (e^t - 1)^{-1} t^{z-1}\, dt$$
converges uniformly on S.

Corollary VII.8.4 (continued 1)

Proof of (b). Let $\varepsilon > 0$ be given. We need to show that there is $M \geq 1$ such that for all $N \geq M$ we have

$$\left|\int_1^\infty (e^t - 1)^{-1} t^{z-1}\,dt - \int_1^N (e^t - 1)^{-1} t^{z-1}\,dt\right|$$

$$= \left|\int_N^\infty (e^t - 1)^{-1} t^{z-1}\,dt\right| < \varepsilon.$$

By Lemma VII.8.3(a), there is $\kappa > 1$ with

$$\left|\int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\,dt\right| < \varepsilon/2$$

for all $\beta > \alpha > \kappa$.

Corollary VII.8.4 (continued 1)

Proof of (b). Let $\varepsilon > 0$ be given. We need to show that there is $M \geq 1$ such that for all $N \geq M$ we have

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$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\, dt \right| < \varepsilon/2$$

for all $\beta > \alpha > \kappa$. So take $M = \kappa + 1$ so that for $N \geq M$ we have

$$\left| \int_N^\infty (e^t - 1)^{-1} t^{z-1}\, dt \right| \leq \varepsilon/2 < \varepsilon$$

and (b) follows (for $\text{Re}(z) \leq A$, as required).

Corollary VII.8.4 (continued 1)

Proof of (b). Let $\varepsilon > 0$ be given. We need to show that there is $M \geq 1$ such that for all $N \geq M$ we have

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and (b) follows (for $\text{Re}(z) \leq A$, as required).

Corollary VII.8.4 (continued 2)

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(a) If $S = \{z \mid a \leq \text{Re}(z) \leq A\}$ where $1 < a < A < \infty$ then the integral
$$\int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$$
converges uniformly on S.

Proof of (a). The proof here is similar to the proof of (b). First, find an M as in the proof of (b) (where we need $\text{Re}(z) \leq A$). Second, find a $\delta > 0$ as guaranteed by Lemma VII.8.3(a) (where we need $a \leq \text{Re}(z)$). □

Proposition VII.8.5

Proposition VII.8.5. For $\text{Re}(z) > 1$

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt.$$

Proof. By the note above, $\int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \text{Re}(z) > 1\}$. So $\zeta(z)\Gamma(z)$ and the integral are both analytic functions on $\{z \mid \text{Re}(z) > 1\}$. By Corollary IV.3.8, if we can show these functions agree on a set with a limit point in $\{z \mid \text{Re}(z) > 1\}$ then the equality is established. We so so for $z \in \{x \in \mathbb{R} \mid x > 1\}$.

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By Lemma VII.8.3, there are α and β where $0 < \alpha < \beta < \infty$ such that:

$$\int_0^\alpha (e^t - 1)^{-1} t^{x-1}\, dt < \frac{\varepsilon}{4} \text{ and } \int_\beta^\infty (e^t - 1)^{-1} t^{x-1}\, dt < \frac{\varepsilon}{4}.$$

For all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} e^{-kt} \leq \sum_{k=1}^{\infty} e^{-kt} = \sum_{k=1}^{\infty} \left(\frac{1}{e^t}\right)^k = \frac{1/e^t}{1 - 1/e^t} = \frac{1}{e^t - 1} = (e^t - 1)^{-1}.$$

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By Lemma VII.8.3, there are α and β where $0 < \alpha < \beta < \infty$ such that:

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For all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^n e^{-kt} \le \sum_{k=1}^\infty e^{-kt} = \sum_{k=1}^\infty \left(\frac{1}{e^t}\right)^k = \frac{1/e^t}{1 - 1/e^t} = \frac{1}{e^t - 1} = (e^t - 1)^{-1}.$$

Proposition VII.8.5 (continued 1)

Proof (continued). So

$$\sum_{k=1}^{n}\left(\int_0^{\alpha} e^{-kt}t^{x-1}\,dt\right) = \int_0^{\alpha}\left(\sum_{k=1}^{n} e^{-kt}t^{x-1}\right)\,dt$$

$$\leq \int_0^{\alpha}(e^t-1)^{-1}t^{x-1}\,dt \leq \frac{\varepsilon}{4}$$

and

$$\sum_{k=1}^{n}\left(\int_{\beta}^{\infty} e^{-kt}t^{x-1}\,dt\right) = \int_{\beta}^{\infty}\left(\sum_{k=1}^{n} e^{-kt}t^{x-1}\right)\,dt$$

$$\leq \int_{\beta}^{\infty}(e^t-1)^{-1}t^{x-1}\,dt < \frac{\varepsilon}{4}$$

for all $a \in \mathbb{N}$. Letting $n \to \infty$ (and replacing index k with n) we get...

Proposition VII.8.5 (continued 2)

Proof (continued).

$$\sum_{n=1}^{\infty}\left(\int_0^{\infty} e^{-nt}t^{x-1}\,dt\right) < \frac{\varepsilon}{4} \text{ and } \sum_{n=1}^{\infty}\left(\int_\beta^{\infty} e^{-nt}t^{x-1}\,dt\right) < \frac{\varepsilon}{4}.$$

Equation (8.2) gives $\zeta(x)\Gamma(x) = \sum_{n=1}^{\infty}\left(\int_0^{\infty} e^{-nt}t^{x-1}\,dx\right)$ so

$$\left|\zeta(z)\Gamma(x) - \int_0^{\infty}(e^t-1)^{-1}t^{x-1}\,dt\right|$$

$$= \left|\sum_{n=1}^{\infty}\int_0^{\infty} e^{-nt}t^{x-1}\,dt - \int_0^{\infty}(e^t-1)^{-1}t^{x-1}\,dt\right|$$

$$= \left|\sum_{n=1}^{\infty}\left(\int_0^{\alpha} e^{-nt}t^{x-1}\,dt + \int_\alpha^{\beta} e^{-nt}t^{x-1}\,dt + \int_\beta^{\infty} e^{-nt}t^{x-1}\,dt\right)\right.$$

$$\left. - \int_\alpha^{\beta}(e^t-1)^{-1}t^{x-1}\,dt - \int_\beta^{\infty}(e^t-1)^{-1}t^{x-1}\,dt\right|$$

Proposition VII.8.5 (continued 3)

Proof (continued).

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left| \sum_{n=1}^{\infty} \left(\int_{\alpha}^{\beta} e^{-nt} t^{x-1}\, dt \right) - \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1}\, dt \right| \qquad (*)$$

(all integrals are positive, so we have absolute convergence of the series and can rearrange). Now $\sum_{n=1}^{\infty} e^{-nt}$ converges uniformly on $[\alpha, \beta]$ (pointwise convergence on a compact set) so

$$\sum_{n=1}^{\infty} \left(\int_{\alpha}^{\beta} e^{-nt} t^{x-1}\, dt \right) = \int_{\alpha}^{\beta} \left(\sum_{n=1}^{\infty} e^{-nt} \right) t^{x-1}\, dt = \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1}\, dt$$

and $(*)$ implies the result. □

Theorem VII.8.C

Lemma VII.8.C. The function $\displaystyle\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $G = \{z \mid \text{Re}(z) > 0\}$.

Proof. Since $\displaystyle\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n$ for some $a_1, a_2, \ldots$ then $\displaystyle\frac{1}{e^t-1} - \frac{1}{t}$ has a limit of $-1/2$ as $t \to 0$. So $\displaystyle\frac{1}{e^t-1} - \frac{1}{t}$ is bounded in a deleted neighborhood of $t = 0$ and so is bounded on $(0, 1]$, say by M.

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$$\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt = \lim_{a\to 0^+} \int_a^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt.$$

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Proof. Since $\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n$ for some $a_1, a_2, \ldots$ then $\frac{1}{e^t - 1} - \frac{1}{t}$ has a limit of $-1/2$ as $t \to 0$. So $\frac{1}{e^t - 1} - \frac{1}{t}$ is bounded in a deleted neighborhood of $t = 0$ and so is bounded on $(0, 1]$, say by M. Let K be a compact subset of $\{z \mid \text{Re}(z) > 0\}$ and let $m = \min\{\text{Re}(z) \mid z \in K\}$. For $z \in K$, consider

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Let $\varepsilon > 0$ and choose $\delta = \min\{(n\varepsilon/M)^{1/m}, 1\}$.

Theorem VII.8.C

Lemma VII.8.C. The function $\displaystyle\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $G=\{z \mid \mathrm{Re}(z)>0\}$.

Proof. Since $\displaystyle\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\sum_{n=1}^{\infty} a_n z^n$ for some $a_1, a_2, \ldots$ then $\displaystyle\frac{1}{e^t-1}-\frac{1}{t}$ has a limit of $-1/2$ as $t\to 0$. So $\displaystyle\frac{1}{e^t-1}-\frac{1}{t}$ is bounded in a deleted neighborhood of $t=0$ and so is bounded on $(0,1]$, say by M. Let K be a compact subset of $\{z \mid \mathrm{Re}(z)>0\}$ and let $m=\min\{\mathrm{Re}(z) \mid z\in K\}$. For $z\in K$, consider

$$\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt=\lim_{a\to 0^+}\int_a^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt.$$

Let $\varepsilon>0$ and choose $\delta=\min\{(n\varepsilon/M)^{1/m},1\}$.

Theorem VII.8.C (continued 1)

Proof (continued). Then for $0 < a < \delta < 1$ we have

$$\left| \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt - \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt \right|$$

$$\begin{aligned}
&= \left| \int_0^a \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt \right| \le \int_0^1 \left| \frac{1}{e^t - 1} - \frac{1}{t} \right| \left| t^{z-1} \right| dt \\
&\le M \int_0^a \left| e^{(z-1)\log t} \right| dt = M \int_0^a e^{(\mathrm{Re}(z)-1)\log t}\, dt \\
&= M \int_0^a t^{\mathrm{Re}(z)-1}\, dt = M \frac{1}{\mathrm{Re}(z)} t^{\mathrm{Re}(z)} \Big|_0^a \\
&= \frac{M}{\mathrm{Re}(z)} a^{\mathrm{Re}(z)} < \frac{M}{\mathrm{Re}(z)} \delta^{\mathrm{Re}(z)} < \frac{M}{m} \delta^m \\
&= \frac{M}{m} \left(\left(\frac{m\varepsilon}{M} \right)^{1/m} \right)^m = \frac{M}{m} \frac{m\varepsilon}{M} = \varepsilon.
\end{aligned}$$

Theorem VII.8.C (continued 2)

Lemma VII.8.C. The function $\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $G=\{z \mid \mathrm{Re}(z)>0\}$.

Proof (continued). So, since δ is independent of $z\in K$, then

$$\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt = \lim_{a\to 0^+}\int_a^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt$$

converges uniformly on K. Since K is an arbitrary compact subset of G, then $\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt$ is uniformly convergent on compact subsets of G. So by Corollary VII.8.B , $\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $\{z \mid \mathrm{Re}(z)>0\}$. □

Theorem VII.8.C (continued 2)

Lemma VII.8.C. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \text{Re}(z) > 0\}$.

Proof (continued). So, since δ is independent of $z \in K$, then

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$$

converges uniformly on K. Since K is an arbitrary compact subset of G, then $\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} \right) t^{z-1}\, dt$ is uniformly convergent on compact subsets of G. So by Corollary VII.8.B , $\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} \right) t^{z-1}\, dt$ is an analytic function on $\{z \mid \text{Re}(z) > 0\}$. □

Theorem VII.8.D

Lemma VII.8.D. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \text{Re}(z) > -1\}$.

Proof. Let K be a compact subset of $G = \{z \mid \text{Re}(z) > -1\}$ and let $m = \min\{\text{Re}(z) \mid z \in K\}$. For $z \in K$ consider

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt.$$

Theorem VII.8.D

Lemma VII.8.D. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > -1\}$.

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$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt.$$

Let $\varepsilon > 0$ and choose $\delta = \min\{(\varepsilon(m+1))^{1/(m+1)}, 1\}$. Then for $0 < a < \delta < 1$ we have

$$\left| \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt \right|$$

Theorem VII.8.D

Lemma VII.8.D. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \text{Re}(z) > -1\}$.

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$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt.$$

Let $\varepsilon > 0$ and choose $\delta = \min\{(\varepsilon(m+1))^{1/(m+1)}, 1\}$. Then for $0 < a < \delta < 1$ we have

$$\left| \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt \right|$$

Theorem VII.8.D (continued 1)

Proof (continued).

$$
\begin{aligned}
= \;& \left| \int_0^a \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt \right| \leq \int_0^1 \left| \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right| |t^{z-1}|\, dt \\
\leq \;& \int_0^a |t||t^{z-1}|\, dt \text{ since } \frac{1}{e^t-1} + \frac{1}{2} \leq ct \text{ for some constant } c \text{ and} \\
& \qquad \text{for } t \in [0,1] \text{ by Exercise VII.8.A} \\
= \;& c \int_0^a |t^z|\, dt - \int_0^a |e^{z \log t}|\, dt = \int_0^a e^{\mathrm{Re}(z) \log t}\, dt \\
= \;& \int_0^a t^{\mathrm{Re}(z)}\, dt = \frac{1}{\mathrm{Re}(z)+1} t^{\mathrm{Re}(z)+1} \Big|_{t=0}^{t=a} \\
= \;& \frac{a^{\mathrm{Re}(z)+1}}{\mathrm{Re}(z)+1} \leq \frac{a^{\mathrm{Re}(z)+1}}{m+1} < \frac{\delta^{\mathrm{Re}(z)+1}}{m+1} < \frac{\delta^{m+1}}{m+1} \\
\leq \;& \frac{((\varepsilon(m+1))^{1/(m+1)})^{m+1}}{m+2} = \frac{\varepsilon(m+1)}{m+1} = \varepsilon.
\end{aligned}
$$

Theorem VII.8.D (continued 2)

Lemma VII.8.D. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > -1\}$.

Proof (continued). So, since δ is independent of $z \in K$, then

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$$

converges uniformly on K. Since K is an arbitrary compact subset of G, then $\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is uniformly convergent on compact subsets of G. So by Corollary VII.8.B, $\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > -1\}$. □

Theorem VII.8.D (continued 2)

Lemma VII.8.D. The function $\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > -1\}$.

Proof (continued). So, since δ is independent of $z \in K$, then

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = \lim_{a \to 0^+} \int_a^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt$$

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Theorem VII.8.E

Lemma VII.8.E. The function $\int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$ is an analytic function on $G = \{z \mid \text{Re}(z) < 1\}$.

Proof. We have $\lim_{t\to\infty} t\left(\frac{1}{t} - \frac{1}{e^t-1}\right) = 1$ by L'hopital's Rule, so $t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)$ is bounded for t sufficiently large, say $t \geq N$. Also, $t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)$ is continuous from $t > 0$, so it is bounded on $[1, \infty)$, say $t\left(\frac{1}{t} - \frac{1}{e^t-1}\right) \leq M$ for $t \geq 1$.

Theorem VII.8.E

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$$\int_1^\infty t\left(\frac{1}{t} - \frac{1}{e^t-1}\right) t^{z-1}\,dt = \lim_{b\to\infty}\left(\int_1^b t\left(\frac{1}{t} - \frac{1}{e^t-1}\right) t^{z-1}\,dt\right).$$

Theorem VII.8.E

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Let $\varepsilon > 0$. Choose $N' = \max\left\{N, \left(\frac{(1-M')\varepsilon}{M}\right)^{1/(\text{Re}(z)-2)}\right\}$.

Theorem VII.8.E

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Let $\varepsilon > 0$. Choose $N' = \max\left\{N, \left(\frac{(1-M')\varepsilon}{M}\right)^{1/(\text{Re}(z)-2)}\right\}$.

Theorem VII.8.E (continued 1)

Proof (continued). Then for $b > N'$ we have

$$\left| \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt - \int_1^b \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt \right|$$

$$\begin{aligned}
&= \left| \int_b^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t t^{z-2}\, dt \right| \leq \int_b^\infty \left| \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t \right| \left| t^{z-2} \right| dt \\
&\leq M \int_b^\infty \left| e^{(z-2)\log t} \right| dt = M \int_b^\infty e^{(\mathrm{Re}(z)-2)\log t}\, dt \\
&\leq M \int_b^\infty t^{\mathrm{Re}(z)-2}\, dt = M \frac{1}{\mathrm{Re}(z)-1} t^{\mathrm{Re}(z)-1} \Big|_b^\infty \\
&= M \frac{1}{\mathrm{Re}(z)-1} \left(\left(\lim_{c\to\infty} c^{\mathrm{Re}(z)-1} \right) - b^{\mathrm{Re}(z)-1} \right) \\
&= M \frac{1}{\mathrm{Re}(z)-1} b^{\mathrm{Re}(z)-1} < \frac{M}{1-\mathrm{Re}(z)} (N')^{\mathrm{Re}(z)-1}
\end{aligned}$$

Theorem VII.8.E (continued 2)

Proof (continued).

$$\leq \frac{M}{1-M'}(N')^{\mathrm{Re}(z)-1} \text{ since } \mathrm{Re}(z) \leq M' \text{ implies} \frac{1}{1-\mathrm{Re}(z)} = \varepsilon$$

$$\leq \frac{M}{1-M'}\left(\left(\frac{(1-M')\varepsilon}{M}\right)^{1/(\mathrm{Re}(z)-1)}\right)^{\mathrm{Re}(z)-1} - \frac{M}{1-M'}\frac{(1-M')\varepsilon}{M} = \varepsilon.$$

So, since N' is independent of $z \in K$, then

$$\int_1^\infty t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt = \lim_{b\to\infty}\left(\int_1^b t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt\right)$$

converges uniformly on K.

Theorem VII.8.E (continued 2)

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So, since N' is independent of $z \in K$, then

$$\int_1^\infty t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt = \lim_{b\to\infty}\left(\int_1^b t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt\right)$$

converges uniformly on K. Since K is an arbitrary compact subset of G, then $\int_1^\infty t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt$ is uniformly convergent on compact subsets of G. So by Theorem VII.8.A/Exercise VII.2.2 $\int_1^\infty t\left(\frac{1}{t} - \frac{1}{e^t-1}\right)t^{z-1}\,dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) < 1\}$. □

Theorem VII.8.E (continued 2)

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$$\leq \frac{M}{1-M'}(N')^{\mathrm{Re}(z)-1} \text{ since } \mathrm{Re}(z) \leq M' \text{ implies} \frac{1}{1-\mathrm{Re}(z)} = \varepsilon$$

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Theorem VII.8.13

Theorem VII.8.13. Riemann's Functional Equation.
For $-1 < \mathrm{Re}(z) < 0$ we have

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin \pi z/2.$$

Proof. If $-1 < \mathrm{Re}(z) < 0$ then

$$\int_1^\infty t^{z-1}\,dt = \frac{1}{z}t^z\bigg|_1^\infty = \lim_{b\to\infty}\frac{1}{z}b^z - \frac{1}{z} = 0 - \frac{1}{z} = -\frac{1}{z}$$

(notice $\lim_{b\to\infty}|b^z| = \lim_{b\to\infty} b^{\mathrm{Re}(z)} = 0$ since $-1 < \mathrm{Re}(z) < 0$).

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(notice $\lim_{b\to\infty}|b^z| = \lim_{b\to\infty} b^{\text{Re}(z)} = 0$ since $-1 < \text{Re}(z) < 0$). So for $-1 < \text{Re}(z) < 0$, we can write (8.8) as

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt + \frac{1}{2}\int_1^\infty t^{z-1}\,dt$$
$$+ \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$$

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$$+ \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$$

Theorem VII.8.13 (continued 1)

Proof (continued). . . . so

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt \text{ for } -1 < \text{Re}(z) < 0. \qquad (8.9)$$

Notice that (see page 32)

$$\cot z = \frac{\cos z}{\sin z} = \frac{(e^{-iz} + e^{-iz})/2}{2(e^{iz} - e^{-iz})/(2i)} = i\frac{1 + e^{-2iz}}{1 - e^{-2iz}}$$

and with $z = -t/(2i) = it/2$ we have $\cot\left(\frac{-1}{2i}t\right) = \cot(it/2) = i\frac{1+e^t}{1-e^t}$ or $\frac{i}{2}\cot(it/2) = -\frac{1}{2}\frac{1+e^t}{1-e^t} = \frac{1}{2}\frac{e^t+1}{e^t-1}$. So we have

$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{2 + e^t - 1}{2(e^t - 1)} = \frac{e^t + 1}{2(e^t - 1)} = \frac{i}{2}\cot(it/2).$$

Theorem VII.8.13 (continued 1)

Proof (continued). ...so

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt \text{ for } -1 < \mathrm{Re}(z) < 0. \qquad (8.9)$$

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$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{2 + e^t - 1}{2(e^t - 1)} = \frac{e^t + 1}{2(e^t - 1)} = \frac{i}{2}\cot(it/2).$$

Theorem VII.8.13 (continued 2)

Proof (continued). By Exercise V.2.8, for $a \notin \mathbb{Z}$ we have
$\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$ so that with $a = it/(2\pi)$ we have
$\pi \cot(it/2) = \frac{2\pi}{it} + \sum_{n=1}^{\infty} \frac{it/\pi}{\frac{-t^2}{4\pi^2} - n^2}$ or

$$\cot(it/2) = \frac{2}{it} + \sum_{n=1}^{\infty} \frac{it}{it^2/4 - \pi^2 n^2} = \frac{2}{it} - 4it \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2}$$

where $it/(2\pi) \notin \mathbb{Z}$ (we'll have t real, so we need only avoid $t = 0$). Then

$$\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{1}{t} = \left(\frac{i}{2}\cot(it/2) - \frac{1}{t}\right)\frac{1}{t}$$

$$= \left(\frac{i}{2}\left(\frac{2}{it} - 4it\sum_{n=1}^{\infty}\frac{1}{t^2 + 4\pi^2 n^2}\right) = \frac{1}{t}\right)\frac{1}{t}$$

Theorem VII.8.13 (continued 2)

Proof (continued). By Exercise V.2.8, for $a \notin \mathbb{Z}$ we have

$\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$ so that with $a = it/(2\pi)$ we have

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$$\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{1}{t} = \left(\frac{i}{2} \cot(it/2) - \frac{1}{t} \right) \frac{1}{t}$$

$$= \left(\frac{i}{2} \left(\frac{2}{it} - 4it \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2} \right) = \frac{1}{t} \right) \frac{1}{t}$$

Theorem VII.8.13 (continued 3)

Proof (continued).

$$= \left(\frac{1}{t} + 2t \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2} - \frac{1}{t} \right) \frac{1}{t} = 2 \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2},$$

so that (8.9) becomes, for $-1 < \text{Re}(z) < 0$,

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}$$

$$= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt + \frac{1}{2} \int_1^{\infty} t^{z-1}\, dt$$

$$+ \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$$

$$= \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt = 2 \int_0^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{t^2 - 4\pi^2 n^2} \right) t^z\, dt.$$

Theorem VII.8.13 (continued 3)

Proof (continued).

$$= \left(\frac{1}{t} + 2t \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2} - \frac{1}{t} \right) \frac{1}{t} = 2 \sum_{n=1}^{\infty} \frac{1}{t^2 + 4\pi^2 n^2},$$

so that (8.9) becomes, for $-1 < \mathrm{Re}(z) < 0$,

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}$$

$$= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt + \frac{1}{2} \int_1^{\infty} t^{z-1}\, dt$$

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Theorem VII.8.13 (continued 4)

Proof (continued). We leave "it to the reader to justify the interchanging of the sum and the integral" (Conway says on page 192; so one must show that the series converges uniformly for $t \in [0, \infty)$). We then have for $-1 < \text{Re}(z) < 0$

$$\zeta(z)\Gamma(z) = 2\sum_{n=1}^{\infty}\left(\int_0^{\infty} \frac{t^z}{t^2 + 4\pi^2 n^2}\, dt\right) = 2\sum_{n=1}^{\infty}\left(\int_0^{\infty} \frac{t^2}{t^2 + (2\pi n)^2}\, dt\right)$$

Let $t = 2\pi n t'$ so that $t' \in [0, \infty)$ and $dt = 2\pi n\, dt'$

$$= 2\sum_{n=1}^{\infty}\int_0^{\infty} \frac{(2\pi n t')^2}{(2\pi n t')^2 + 2\pi n)^2} 2\pi b\, dt'$$

$$= 2\sum_{n=1}^{\infty}(2\pi n)^{z-1}\left(\int_0^{\infty} \frac{(t')^z}{(t')^2 + 1}\, dt'\right)$$

$$= 2\sum_{n=1}^{\infty}(2\pi n)^{z-1}\left(\int_0^{\infty} \frac{t^z}{t^2 + 1}\, dt\right)$$

Theorem VII.8.13 (continued 4)

Proof (continued). We leave "it to the reader to justify the interchanging of the sum and the integral" (Conway says on page 192; so one must show that the series converges uniformly for $t \in [0, \infty)$). We then have for $-1 < \text{Re}(z) < 0$

$$\begin{aligned}
\zeta(z)\Gamma(z) &= 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{t^z}{t^2+4\pi^2 n^2}\,dt\right) = 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{t^2}{t^2+(2\pi n)^2}\,dt\right) \\
&\quad \text{Let } t = 2\pi n t' \text{ so that } t' \in [0,\infty) \text{ and } dt = 2\pi n\,dt' \\
&= 2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{(2\pi n t')^2}{(2\pi n t')^2+2\pi n)^2}2\pi b\,dt' \\
&= 2\sum_{n=1}^{\infty}(2\pi n)^{z-1}\left(\int_0^{\infty}\frac{(t')^z}{(t')^2+1}\,dt'\right) \\
&= 2\sum_{n=1}^{\infty}(2\pi n)^{z-1}\left(\int_0^{\infty}\frac{t^z}{t^2+1}\,dt\right)
\end{aligned}$$

Theorem VII.8.13 (continued 4)

Proof (continued).

$$\begin{aligned}\zeta(z)\Gamma(z) &= 2(2\pi)^{z-2}\left(\int_0^\infty \frac{t^z}{t^2+1}\,dt\right)\sum_{n=1}^\infty n^{z-1} \\ &= 2(2\pi)^{z-1}\zeta(1-z)\int_0^\infty \frac{t^z}{t^2+1}\,dt \qquad (8.11)\end{aligned}$$

since $\text{Re}(1-z) > 1$ and by definition $\zeta(1-z) = \sum_{n=1}^\infty n^{-(1-z)} = \sum_{n=1}^\infty n^{z-1}$. For $z \in \mathbb{R}$ where $-1 < x < 0$, by Example V.2.12 we have

$$\int_0^\infty \frac{t^x}{t^2+1}\,dt = \frac{1}{2}\int_0^\infty \frac{s^{(z-1)/2}}{s+1}\,ds = \frac{\pi}{2}\cos\frac{\pi(1-x)}{2} = \frac{\pi}{2}\sec\frac{\pi x}{2}. \quad (8.12)$$

Theorem VII.8.13 (continued 4)

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By Exercise VII.7.2,

$$\frac{1}{\Gamma(x)} = \frac{\Gamma(1-x)}{\pi}\sin(\pi x) - \frac{\Gamma(1-x)}{\pi}2\sin(\pi x/2)\cos(\pi x/2),$$

so from (8.11) we have (for $z = x$)...

Theorem VII.8.13 (continued 4)

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Theorem VII.8.13 (continued 5)

Proof (continued).

$$\zeta(x) = \frac{1}{\Gamma(x)} 2(2\pi)^{x-1}\zeta(1-x) \int_0^\infty \frac{t^x}{t^2+1}\,dt$$

$$= \frac{1}{\Gamma(x)} 2(2\pi)^{x-1}\zeta(1-x)\frac{1}{2}\pi\sec(\pi x/2)$$

$$= \left(\frac{\Gamma(1-x)}{\pi} 2\sin(\pi x/2)\cos(\pi x/2)\right)\left(2(2\pi)^{x-1}\zeta(1-x)\frac{1}{2}\pi\sec(\pi x/2)\right)$$

$$= 2(2\pi)^{x-1}\Gamma(1-x)\zeta(1-x)\sin(\pi x/2) \text{ for } -1 < x < 0.$$

So we have defined $\zeta(z)$ and $2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\pi z/2)$ as analytic functions in $-1 < \mathrm{Re}(z) < 0$. Since they are equal on $-1 < x < 0$ then, by Corollary IV.3.8, they are equal on $-1 < \mathrm{Re}(z) < 0$. □

Theorem VII.8.13 (continued 5)

Proof (continued).

$$\zeta(x) = \frac{1}{\Gamma(x)} 2(2\pi)^{x-1}\zeta(1-x)\int_0^\infty \frac{t^x}{t^2+1}\,dt$$

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So we have defined $\zeta(z)$ and $2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\pi z/2)$ as analytic functions in $-1 < \mathrm{Re}(z) < 0$. Since they are equal on $-1 < x < 0$ then, by Corollary IV.3.8, they are equal on $-1 < \mathrm{Re}(z) < 0$. □

Theorem VII.8.17

Theorem VII.8.17. Euler's Theorem.
If $\mathrm{Re}(z) > 1$ then

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$

where $\{p_n\}$ is the sequence of prime numbers.

Proof. For each $n \in \mathbb{N}$ we have $\sum_{m=0}^{\infty} p_n^{-mz} = \frac{1}{1-p_n^{-z}}$ since this is a geometric series with ration p_n^{-z} (and $|p_n^{-z}| = p_n^{\mathrm{Re}(-z)} < 1$). For each $n \in \mathbb{N}$,

$$\prod_{k=1}^{n} \frac{1}{1 - p_k^{-z}} = \prod_{k=1}^{n} \left(\sum_{m=0}^{\infty} p_n^{-mz} \right).$$

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$$\prod_{k=1}^{n} \frac{1}{1 - p_k^{-z}} = \prod_{k=1}^{n} \left(\sum_{m=0}^{\infty} p_n^{-mz} \right).$$

Now the n series on the right hand side converge absolutely (they are geometric series) and so can be rearranged. So the right hand side includes all elements of $\mathbb{N}$ to the z power which are products of powers of the primes $p_1, p_2, \ldots, p_n$.

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Theorem VII.8.17 (continued)

Proof (continued). Denote these natural numbers as $n_1, n_2, \ldots$. Notice that each such n_j appears only once by the Fundamental Theorem of Arithmetic. So

$$\prod_{k=1}^{n} \frac{1}{1 - p_k^{-z}} = \sum_{j=1}^{\infty} n_j^{-z}.$$

So letting $n \to \infty$ then all primes are included and so all natural numbers are included so that

$$\prod_{k=1}^{\infty} \frac{1}{a - p_k^{-z}} = \prod_{k=1}^{\infty} \left(\sum_{m=0}^{\infty} p_n^{-mz} \right) = \sum_{j=1}^{\infty} n_j^{-z} = \zeta(z).$$

□

Theorem VII.8.17 (continued)

Proof (continued). Denote these natural numbers as $n_1, n_2, \ldots$. Notice that each such n_j appears only once by the Fundamental Theorem of Arithmetic. So

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