

Theorem VIII.2.2

Complex Analysis

Chapter VIII. Runge's Theorem

VIII.2. Simple Connectedness—Proofs of Theorems



Theorem VIII.2.2. Let G be an open connected subset of \mathbb{C} . The the following are equivalent:

- (a) G is simply connected;
 - (b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} \setminus G$;
 - (c) $\mathbb{C}_\infty \setminus G$ is connected;
 - (d) For any $f \in H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;
 - (e) For any $f \in H(G)$ and any closed rectifiable curve γ in G , $\int_\gamma f = 0$;
 - (f) Every function $f \in H(G)$ has a primitive;
- (continued...)

0

Complex Analysis

August 30, 2017

1 / 14

0

Complex Analysis

August 30, 2017

3 / 14

Theorem VIII.2.2

Theorem VIII.2.2 (continued 1)

Theorem VIII.2.2. Let G be an open connected subset of \mathbb{C} . The the following are equivalent:

- (g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;
- (h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;
- (i) G is homeomorphic to the unit disk;
- (j) If $u : G \rightarrow \mathbb{R}$ is harmonic then there is a harmonic function $v : G \rightarrow \mathbb{R}$ such that $f = u + iv$ is analytic on G .

Proof. We prove the implications as follows: (a) \Rightarrow (b) \Rightarrow (c) $\Rightarrow \dots \Rightarrow$ (i) \Rightarrow (a), and then (h) \Rightarrow (j) \Rightarrow (g).

Theorem VIII.2.2

Theorem VIII.2.2 (continued 2)

- (a) G is simply connected;
- (b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} \setminus G$;

Proof (continued). (a) \Rightarrow (b). Let γ be a closed rectifiable curve in G and $a \in \mathbb{C} \setminus G$. Then $(z - a)^{-1}$ is analytic on G and

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma (z - a)^{-1} dz = 0$$

by Cauchy's Theorem—Fourth Version (Theorem IV.6.15).

(b) \Rightarrow (c). ASSUME $\mathbb{C}_\infty \setminus G$ is not connected. Then $\mathbb{C}_\infty \setminus G = A \cup B$ where A and B are disjoint, nonempty closed subsets of \mathbb{C}_∞ (from the definition of connected metric space, Definition II.2.1; A and B are both open and closed in $\mathbb{C}_\infty \setminus G$).

0

Complex Analysis

August 30, 2017

4 / 14

0

Complex Analysis

August 30, 2017

5 / 14

Theorem VIII.2.2 (continued 3)

Proof (continued). Since ∞ is in either A or B , say $\infty \in B$, then A must be a compact subset of \mathbb{C} (A is closed in \mathbb{C}_∞ and \mathbb{C}_∞ is compact [see “Compactness of \mathbb{C}_∞ Theorem” in the supplement to Section II.4, “The Extended Complex Plane”] so A is compact in \mathbb{C}_∞ ; every open cover of A in \mathbb{C} yields an open cover of A in \mathbb{C}_∞ by the “Topologies on \mathbb{C}_∞ Theorem” in the supplement, so A is compact in \mathbb{C} also). Since $\mathbb{C}_\infty \setminus G = A \cup B$ then $G_1 = G \cup A = \mathbb{C}_\infty \setminus B$ is an open set in \mathbb{C} and contains set A . So by Proposition VIII.1.1 (since G_1 is open, A is compact, and $G_1 \setminus A = G$ is a region) there are a finite number of polygons $\gamma_1, \gamma_2, \dots, \gamma_n$ in $G_1 \setminus A = G$ such that for every analytic function f on G , we have

$$f(z) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw \text{ for all } z \in A. \text{ So with } f(z) \equiv 1 \text{ we have}$$

$$1 = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{1}{w-z} dw = \sum_{k=1}^m N(\gamma_k; z) \text{ for all } z \in A.$$

0

Complex Analysis

August 30, 2017

6 / 14

Theorem VIII.2.2

Theorem VIII.2.2 (continued 5)

- (e) For any $f \in H(G)$ and any closed rectifiable curve γ in G ,
 $\int_\gamma f = 0$;
- (f) Every function $f \in H(G)$ has a primitive;

Proof (continued). Since each polynomial is analytic on \mathbb{C} and $\gamma \sim 0$ in \mathbb{C} then $\int_\gamma p_n = 0$ for $n \in \mathbb{N}$ by Cauchy’s Theorem–Fourth Version (Theorem IV.6.15). So, by the uniform convergence on $\{\gamma\}$,
 $\int_\gamma f = \lim_{n \rightarrow \infty} \int_\gamma p_n = 0$ by Lemma IV.2.7.

(e) \Rightarrow (f). Fix $a \in G$. Let $z \in G$. For γ_1 and γ_2 any rectifiable curves in G from a to z we have $\gamma_1 - \gamma_2$ is a closed rectifiable curve in G . So by hypothesis (e) we have $0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$ and so $\int_{\gamma_1} f = \int_{\gamma_2} f$. So define $F(z) = \int_\gamma f$ where γ is any rectifiable curve in G from a to z (F is well-defined, that is independent of γ , as just shown). As shown in the proof of Corollary IV.6.16, $F' = f$ on G and so f has a primitive F on G .

0

Complex Analysis

August 30, 2017

8 / 14

Theorem VIII.2.2

Theorem VIII.2.2 (continued 4)

- (c) $\mathbb{C}_\infty \setminus G$ is connected;
- (d) For any $f \in H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;
- (e) For any $f \in H(G)$ and any closed rectifiable curve γ in G ,
 $\int_\gamma f = 0$;

Proof (continued). But since $n(\gamma_k; z) \in \mathbb{N} \cup \{0\}$, then for some $1 \leq k \leq m$ we have $n(\gamma_k) = 1$, CONTRADICTING (b) that all such winding numbers for $a \in A \subset \mathbb{C} \setminus G$ are 0. So the assumption that $\mathbb{C}_\infty \setminus G$ is not connected is false and $\mathbb{C}_\infty \setminus G$ is therefore connected.

(c) \Rightarrow (d). This is Corollary VIII.1.15.

(d) \Rightarrow (e). Let γ be a closed rectifiable curve in G , let $f \in H(G)$ (i.e., f is analytic on G), and let $\{p_n\}$ be a sequence of polynomials such that $f = \lim_{n \rightarrow \infty} p_n$ in $H(G)$. So $p_n \rightarrow f$ uniformly on compact subsets of G by Proposition VII.1.10(b). So $p_n \rightarrow f$ uniformly on compact set $\{\gamma\}$.

0

Complex Analysis

August 30, 2017

7 / 14

Theorem VIII.2.2

Theorem VIII.2.2 (continued 6)

- (f) Every function $f \in H(G)$ has a primitive;
- (g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;
- (h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

Proof (continued). (f) \Rightarrow (g). If $f(z) \neq 0$ for all $z \in G$ then f'/f is analytic on G . By hypothesis (f) there is a function F with $F' = f'/f$. It is shown in the proof of Corollary IV.6.17 that $f(z) = \exp(F(z) + c)$ for some constant $c \in \mathbb{C}$ (in the interior of the proof of Corollary IV.6.17, $g_1 = F$).

(g) \Rightarrow (h). By hypothesis (g), $f(z) = \exp(g(z))$. So consider $\exp(g(z)/2)$. We have $(\exp(g(z)/2))^2 = \exp(g(z)) = f(z)$ and so $\exp(g(z)/2)$ (in the notation of (g)) is the desired function (also denoted as “ g ” in the notation of (h)).

0

Complex Analysis

August 30, 2017

9 / 14

Theorem VIII.2.2 (continued 7)

- (a) G is simply connected;
- (h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;
- (i) G is homeomorphic to the unit disk;
- Proof (continued).** (h) \Rightarrow (i). If $G = \mathbb{C}$ then $f(z) = z/(1 + |z|)$ is a homeomorphism from $G = \mathbb{C}$ to the unit disk (regardless of hypothesis (h)). If $G \neq \mathbb{C}$ and (h) holds then every nonvanishing analytic function on G has an analytic square root. So the hypotheses of Lemma VII.4.3 are satisfied. So by Lemma VII.4.3 there exists an analytic one to one function h from G onto the unit disk. By Corollary IV.7.6 (a corollary to the Open Mapping Theorem) h^{-1} is analytic (and so continuous). Therefore f is a homeomorphism and (i) follows.
- (i) \Rightarrow (a). Let $h : G \rightarrow D = \{z \mid |z| < 1\}$ be a homeomorphism and let γ be a closed curve in G (not assumed to be rectifiable).

0

Complex Analysis

August 30, 2017

10 / 14

Theorem VIII.2.2 (continued 9)

- (h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;
- (j) If $u : G \rightarrow R$ is harmonic then there is a harmonic function $v : G \rightarrow R$ such that $f = u + iv$ is analytic on G .

Proof (continued). (h) \Rightarrow (j). Suppose $G \neq \mathbb{C}$. Since (h) holds then G is simply connected (since (h) \Rightarrow (i) \Rightarrow (a)). So by the Riemann Mapping Theorem (Theorem VII.4.2) there is analytic function h on G such that h is one to one and $h(G) = D$. If $u : G \rightarrow R$ is a harmonic function, consider $u_1 = u \circ h^{-1}$. By Corollary IV.7.6, h^{-1} is analytic. In Exercise VIII.2.A, one shows that u_1 is harmonic in D (using the Chain Rule and the Cauchy-Riemann equations). By Theorem III.2.30 there is a harmonic conjugate of u_1 , $v_1 : D \rightarrow R$, such that $f_1 = u_1 + iv_1$ is analytic on D . Let $f = f_1 \circ h$. Then f is analytic on G and the real part of $f = f_1 \circ h = (u_1 + iv_1) \circ h = (u \circ h^{-1} + iv_1) \circ h = u + iv_1 \circ h$ is u . So $v = \text{Im}(f) = v_1 \circ h$ is a harmonic conjugate of u . So (j) holds on G .

0

Complex Analysis

August 30, 2017

12 / 14

Theorem VIII.2.2 (continued 8)

Proof (continued). Then $\sigma(s) = h(\gamma(a))$ is a closed curve in D . Since D is simply connected then $\sigma \sim 0$ and there is a homotopy Λ mapping σ to a constant (we map σ to $0 \in D$). Thus, there is a continuous function $\Lambda : I^2 \rightarrow D$ such that $\Lambda(s, 0) = \sigma(s)$ for $s \in I$, $\Lambda(s, 1) = 0$ for $s \in I$, and $\Lambda(0, t) = \Lambda(1, t)$ for $t \in I$. That is, $\Lambda(s, t)$ “starts” (when $t = 0$) at $\sigma(s)$, “ends” (when $t = 1$) at the point 0 (since D is simply connected and so $\sigma \sim 0$), and is a closed curve for $t \in I$ (so the endpoints are the same for all $t \in I$, $\Lambda(0, t) = \Lambda(1, t)$). Define $\Gamma = h^{-1} \circ \Lambda$. Then Γ is continuous and $\Gamma : I^2 \rightarrow G$. Also, $\Gamma(s, 0) = h^{-1} \circ \Lambda(s, 0) = h^{-1} \circ \sigma(s) = \gamma(s)$ for $s \in I$ (so Γ “starts” at at $\gamma(s)$), $\Gamma(s, 1) = h^{-1} \circ \Lambda(s, 1) = h^{-1}(0)$ for $s \in I$ (so Γ “ends” at the constant $h^{-1}(0) \in G$), and $\Gamma(0, t) = h^{-1} \circ \Lambda(0, t) = h^{-1} \circ \Lambda(1, t) = \Gamma(1, t)$ for $t \in I$ (so the endpoints are the same for all $t \in I$). That is, Γ is a homotopy between γ and the constant $h^{-1}(0)$; so $\gamma \sim 0$. Since γ is an arbitrary closed curve in G , then G is simply connected and (a) holds.

0

Complex Analysis

August 30, 2017

11 / 14

Theorem VIII.2.2 (continued 10)

- (g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;
- (j) If $u : G \rightarrow R$ is harmonic then there is a harmonic function $v : G \rightarrow R$ such that $f = u + iv$ is analytic on G .

Proof (continued). If $G = \mathbb{C}$ then we take $h : \mathbb{C} \rightarrow D$ as $h(z) = z/(1 + |z|)$ and repeat the process above applying Theorem II.2.30.

(j) \Rightarrow (g). Suppose $f : G \rightarrow \mathbb{C}$ is analytic and never vanishes on G , and let $u = \text{Re}(f)$ and $v = \text{Im}(f)$. If $U : G \rightarrow R$ is defined by $U(x, y) = \log |f(x + iy)| = \log(u(x, y)^2 + v(x, y)^2)^{1/2}$ then “a computation” shows that U is harmonic. By hypothesis (j) there is a harmonic function V on G such that $g = U + iV$ is analytic on G . Let $h(z) = \exp(g(z))$. Then h is analytic on F , h never vanishes on G , and $|f(z)/h(z)| = 1$ for all $z \in G$ since

$$|h| = |\exp g| = |\exp(U + iV)| = \exp U = \exp(\log |f|) = |f|.$$

0

Complex Analysis

August 30, 2017

13 / 14

Theorem VIII.2.2 (continued 11)

Proof (continued). So f/h is analytic on G . By Exercise VI.1.6 there is a constant c of modulus 1 such that

$$f(z) = ch(z) = z \exp(h(z)) = \exp(g(z) + c_1) \text{ where } c = \exp(c_1).$$

So $g(z) + c_1$ is a branch of $\log f(z)$ and (g) holds. □