Complex Analysis

Chapter VIII. Runge's Theorem

VIII.2. Simple Connectedness—Proofs of Theorems

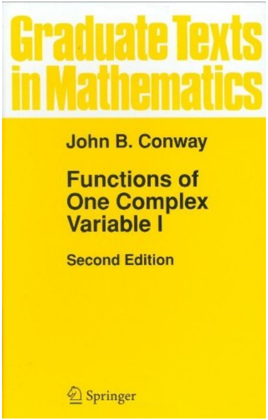

Table of contents

1 Theorem VIII.2.2

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(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} \setminus G$;

(c) $\mathbb{C}_\infty \setminus G$ is connected;

(d) For any $f \in H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;

(e) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;

(f) Every function $f \in H(G)$ has a primitive;

(continued...)

Theorem VIII.2.2 (continued 1)

Theorem VIII.2.2. Let G be an open connected subset of $\mathbb{C}$. The the following are equivalent:

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

(i) G is homeomorphic to the unit disk;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof. We prove the implications as follows: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow \cdots \Rightarrow$ (i) $\Rightarrow$ (a), and then (h) $\Rightarrow$ (j) $\Rightarrow$ (g).

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Theorem VIII.2.2 (continued 2)

(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} \setminus G$;

Proof (continued). (a) $\Rightarrow$ (b). Let γ be a closed rectifiable curve in G and $a \in \mathbb{C} \setminus G$. Then $(z-a)^{-1}$ is analytic on G and

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma (z-a)^{-1}\,dz = 0$$

by Cauchy's Theorem—Fourth Version (Theorem IV.6.15).

(b) $\Rightarrow$ (c). ASSUME $\mathbb{C}_\infty \setminus G$ is not connected. Then $\mathbb{C}_\infty \setminus G = A \cup B$ where A and B are disjoint, nonempty closed subsets of $\mathbb{C}_\infty$ (from the definition of connected metric space, Definition II.2.1; A and B are both open and closed in $\mathbb{C}_\infty \setminus G$).

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Theorem VIII.2.2 (continued 3)

Proof (continued). Since ∞ is in either A or B, say $\infty \in B$, then A must be a compact subset of $\mathbb{C}$ (A is closed in $\mathbb{C}_\infty$ and $\mathbb{C}_\infty$ is compact [see "Compactness of $\mathbb{C}_\infty$ Theorem" in the supplement to Section II.4, "The Extended Complex Plane"] so A is compact in $\mathbb{C}_\infty$; every open cover of A in $\mathbb{C}$ yields an open cover of A in $\mathbb{C}_\infty$ by the "Topologies on $\mathbb{C}_\infty$ Theorem" in the supplement, so A is compact in $\mathbb{C}$ also). Since $\mathbb{C}_\infty \setminus G = A \cup B$ then $G_1 = G \cup A = \mathbb{C}_\infty \setminus B$ is an open set in $\mathbb{C}$ and contains set A. So by Proposition VIII.1.1 (since G_1 is open, A is compact, and $G_1 \setminus A = G$ is a region) there are a finite number of polygons $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $G_1 \setminus A = G$ such that for every analytic function f on G, we have $f(z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw$ for all $z \in A$.

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$$1 = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{1}{w-z}\, dw = \sum_{k=1}^{m} N(\gamma_k; z) \text{ for all } z \in A.$$

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Theorem VIII.2.2 (continued 4)

(c) $\mathbb{C}_\infty \setminus G$ is connected;

(d) For any $f \in H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;

(e) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;

Proof (continued). But since $n(\gamma_k; z) \in \mathbb{N} \cup \{0\}$, then for some $1 \le k \le m$ we have $n(\gamma_k) = 1$, CONTRADICTING (b) that all such winding numbers for $a \in A \subset \mathbb{C} \setminus G$ are 0. So the assumption that $\mathbb{C}_\infty \setminus G$ is not connected is false and $\mathbb{C}_\infty \setminus G$ is therefore connected.

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(d) $\Rightarrow$ (e). Let γ be a closed rectifiable curve in G, let $f \in H(G)$ (i.e., f is analytic on G), and let $\{p_n\}$ be a sequence of polynomials such that $f = \lim_{n\to\infty} p_n$ in $H(G)$. So $p_n \to f$ uniformly on compact subsets of G by Proposition VII.1.10(b). So $p_n \to f$ uniformly on compact set $\{\gamma\}$.

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Theorem VIII.2.2 (continued 5)

(e) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;

(f) Every function $f \in H(G)$ has a primitive;

Proof (continued). Since each polynomial is analytic on $\mathbb{C}$ and $\gamma \sim 0$ in $\mathbb{C}$ then $\int_\gamma p_n = 0$ for $n \in \mathbb{N}$ by Cauchy's Theorem–Fourth Version (Theorem IV.6.15). So, by the uniform convergence on $\{\gamma\}$, $\int_\gamma f = \lim_{n\to\infty} \int_\gamma p_n = 0$ by Lemma IV.2.7.

(e) $\Rightarrow$ (f). Fix $a \in G$. Let $z \in G$. For γ_1 and γ_2 any rectifiable curves in G from a to z we have $\gamma_1 - \gamma_2$ is a closed rectifiable curve in G. So by hypothesis (e) we have $0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$ and so $\int_{\gamma_1} f = \int_{\gamma_2} f$.

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Theorem VIII.2.2 (continued 6)

(f) Every function $f \in H(G)$ has a primitive;

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

Proof (continued). (f) $\Rightarrow$ (g). If $f(z) \neq 0$ for all $z \in G$ then f'/f is analytic on G. By hypothesis (f) there is a function F with $F' = f'/f$. It is shown in the proof of Corollary IV.6.17 that $f(z) = \exp(F(z) + c)$ for some constant $c \in \mathbb{C}$ (in the interior of the proof of Corollary IV.6.17, $g_1 = F$).

(g) $\Rightarrow$ (h). By hypothesis (g), $f(z) = \exp(g(z))$. So consider $\exp(g(z)/2)$. We have $(\exp(g(z)/2))^2 = \exp(g(z)) = f(z)$ and so $\exp(g(z)/2)$ (in the notation of (g)) is the desired function (also denoted as "g" in the notation of (h)).

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(a) G is simply connected;

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(i) G is homeomorphic to the unit disk;

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(i) $\Rightarrow$ (a). Let $h : G \to D = \{z \mid |z| < 1\}$ be a homeomorphism and let γ be a closed curve in G (not assumed to be rectifiable).

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Theorem VIII.2.2 (continued 8)

Proof (continued). Then $\sigma(s) = h(\gamma(a))$ is a closed curve in D. Since D is simply connected then $\sigma \sim 0$ and there is a homotopy Λ mapping σ to a constant (we map σ to $0 \in D$). Thus, there is a continuous function $\Lambda : I^2 \to D$ such that $\Lambda(s, 0) = \sigma(s)$ for $s \in I$, $\Lambda(s, 1) = 0$ for $s \in I$, and $\Lambda(0, t) = \Lambda(1, t)$ for $t \in I$. That is, $\Lambda(s, t)$ "starts" (when $t = 0$) at $\sigma(s)$, "ends" (when $t = 1$) at the point 0 (since D is simply connected and so $\sigma \sim 0$), and is a closed curve for $t \in I$ (so the endpoints are the same for all $t \in I$, $\Lambda(0, t) = \Lambda(1, t)$).

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Proof (continued). Then $\sigma(s) = h(\gamma(a))$ is a closed curve in D. Since D is simply connected then $\sigma \sim 0$ and there is a homotopy Λ mapping σ to a constant (we map σ to $0 \in D$). Thus, there is a continuous function $\Lambda : I^2 \to D$ such that $\Lambda(s,0) = \sigma(s)$ for $s \in I$, $\Lambda(s,1) = 0$ for $s \in I$, and $\Lambda(0,t) = \Lambda(1,t)$ for $t \in I$. That is, $\Lambda(s,t)$ "starts" (when $t = 0$) at $\sigma(s)$, "ends" (when $t = 1$) at the point 0 (since D is simply connected and so $\sigma \sim 0$), and is a closed curve for $t \in I$ (so the endpoints are the same for all $t \in I$, $\Lambda(0,t) = \Lambda(1,t)$). Define $\Gamma = h^{-1} \circ \Lambda$. Then Γ is continuous and $\Gamma : I^2 \to G$. Also, $\Gamma(s,0) = h^{-1} \circ \Lambda(s,0) - f^{-1} \circ \sigma(s) = \gamma(s)$ for $s \in I$ (so Γ "starts" at at $\gamma(s)$), $\Gamma(s,1) = h^{-1} \circ \Lambda(s,1) = h^{-1}(0)$ for $s \in I$ (so Γ "ends" at the constant $h^{-1}(0) \in G$), and $\Gamma(0,t) = h^{-1} \circ \Lambda(0,t) = h^{-1}\Lambda(1,t) = \Gamma(1,t)$ for $t \in I$ (so the endpoints are the same for all $t \in I$). That is, Γ is a homotopy between γ and the constant $h^{-1}(0)$; so $\gamma \sim 0$. Since γ is an arbitrary closed curve in G, then G is simply connected and (a) holds.

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Proof (continued). Then $\sigma(s) = h(\gamma(a))$ is a closed curve in D. Since D is simply connected then $\sigma \sim 0$ and there is a homotopy Λ mapping σ to a constant (we map σ to $0 \in D$). Thus, there is a continuous function $\Lambda : I^2 \to D$ such that $\Lambda(s,0) = \sigma(s)$ for $s \in I$, $\Lambda(s,1) = 0$ for $s \in I$, and $\Lambda(0,t) = \Lambda(1,t)$ for $t \in I$. That is, $\Lambda(s,t)$ "starts" (when $t = 0$) at $\sigma(s)$, "ends" (when $t = 1$) at the point 0 (since D is simply connected and so $\sigma \sim 0$), and is a closed curve for $t \in I$ (so the endpoints are the same for all $t \in I$, $\Lambda(0,t) = \Lambda(1,t)$). Define $\Gamma = h^{-1} \circ \Lambda$. Then Γ is continuous and $\Gamma : I^2 \to G$. Also, $\Gamma(s,0) = h^{-1} \circ \Lambda(s,0) - f^{-1} \circ \sigma(s) = \gamma(s)$ for $s \in I$ (so Γ "starts" at at $\gamma(s)$), $\Gamma(s,1) = h^{-1} \circ \Lambda(s,1) = h^{-1}(0)$ for $s \in I$ (so Γ "ends" at the constant $h^{-1}(0) \in G$), and $\Gamma(0,t) = h^{-1} \circ \Lambda(0,t) = h^{-1}\Lambda(1,t) = \Gamma(1,t)$ for $t \in I$ (so the endpoints are the same for all $t \in I$). That is, Γ is a homotopy between γ and the constant $h^{-1}(0)$; so $\gamma \sim 0$. Since γ is an arbitrary closed curve in G, then G is simply connected and (a) holds.

Theorem VIII.2.2 (continued 9)

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). (h) $\Rightarrow$ (j). Suppose $G \neq \mathbb{C}$. Since (h) holds then G is simply connected (since (h) $\Rightarrow$ (i) $\Rightarrow$ (a)). So by the Riemann Mapping Theorem (Theorem VII.4.2) there is analytic function h on G such that h is one to one and $h(G) = D$. If $u : G \to R$ is a harmonic function, consider $u_1 = u \circ h^{-1}$. By Corollary IV.7.6, h^{-1} is analytic. In Exercise VIII.2.A, one shows that u_1 is harmonic in D (using the Chain Rule and the Cauchy-Riemann equations). By Theorem III.2.30 there is a harmonic conjugate of $u_1, v_1 : D \to R$, such that $f_1 = u_1 + iv_1$ is analytic on D.

Theorem VIII.2.2 (continued 9)

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). (h) $\Rightarrow$ (j). Suppose $G \neq \mathbb{C}$. Since (h) holds then G is simply connected (since (h) $\Rightarrow$ (i) $\Rightarrow$ (a)). So by the Riemann Mapping Theorem (Theorem VII.4.2) there is analytic function h on G such that h is one to one and $h(G) = D$. If $u : G \to R$ is a harmonic function, consider $u_1 = u \circ h^{-1}$. By Corollary IV.7.6, h^{-1} is analytic. In Exercise VIII.2.A, one shows that u_1 is harmonic in D (using the Chain Rule and the Cauchy-Riemann equations). By Theorem III.2.30 there is a harmonic conjugate of $u_1, v_1 : D \to R$, such that $f_1 = u_1 + iv_1$ is analytic on D. Let $f = f_1 \circ h$. Then f is analytic on G and the real part of $f = f_1 \circ h = (u_1 + iv_1) \circ h = (u \circ h^{-1} + iv_1) \circ h = u + iv_1 \circ h$ is u. So $v = \mathrm{Im}(f) = v_1 \circ h$ is a harmonic conjugate of u. So (j) holds on G.

Theorem VIII.2.2 (continued 9)

(h) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = (g(z))^2$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). (h) $\Rightarrow$ (j). Suppose $G \neq \mathbb{C}$. Since (h) holds then G is simply connected (since (h) $\Rightarrow$ (i) $\Rightarrow$ (a)). So by the Riemann Mapping Theorem (Theorem VII.4.2) there is analytic function h on G such that h is one to one and $h(G) = D$. If $u : G \to R$ is a harmonic function, consider $u_1 = u \circ h^{-1}$. By Corollary IV.7.6, h^{-1} is analytic. In Exercise VIII.2.A, one shows that u_1 is harmonic in D (using the Chain Rule and the Cauchy-Riemann equations). By Theorem III.2.30 there is a harmonic conjugate of $u_1, v_1 : D \to R$, such that $f_1 = u_1 + iv_1$ is analytic on D. Let $f = f_1 \circ h$. Then f is analytic on G and the real part of $f = f_1 \circ h = (u_1 + iv_1) \circ h = (u \circ h^{-1} + iv_1) \circ h = u + iv_1 \circ h$ is u. So $v = \text{Im}(f) = v_1 \circ h$ is a harmonic conjugate of u. So (j) holds on G.

Theorem VIII.2.2 (continued 10)

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). If $G = \mathbb{C}$ then we take $h : \mathbb{C} \to D$ as $h(z) = z/(1 + |z|)$ and repeat the process above applying Theorem II.2.30.

(j) $\Rightarrow$ (g). Suppose $f : G \to \mathbb{C}$ is analytic and never vanishes on G, and let $u = \text{Re}(f)$ and $v = \text{Im}(f)$. If $U : G \to R$ is defined by $U(x, y) = \log |f(x + iy)| = \log(u(x, y)^2 + v(x, y)^2)^{1/2}$ then "a computation" shows that U is harmonic.

Theorem VIII.2.2 (continued 10)

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). If $G = \mathbb{C}$ then we take $h : \mathbb{C} \to D$ as $h(z) = z/(1 + |z|)$ and repeat the process above applying Theorem II.2.30.

(j) $\Rightarrow$ (g). Suppose $f : G \to \mathbb{C}$ is analytic and never vanishes on G, and let $u = \text{Re}(f)$ and $v = \text{Im}(f)$. If $U : G \to R$ is defined by $U(x, y) = \log |f(x + iy)| = \log(u(x, y)^2 + v(x, y)^2)^{1/2}$ then "a computation" shows that U is harmonic. By hypothesis (j) there is a harmonic function V on G such that $g = U + iV$ is analytic on G. Let $h(z) = \exp(g(z))$. Then h is analytic on F, h never vanishes on G, and $|f(z)/h(z)| = 1$ for all $z \in G$ since

$$|h| = |\exp g| = |\exp(U + iV)| = \exp U = \exp(\log |f|) = |f|.$$

Theorem VIII.2.2 (continued 10)

(g) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;

(j) If $u : G \to R$ is harmonic then there is a harmonic function $v : G \to R$ such that $f = u + iv$ is analytic on G.

Proof (continued). If $G = \mathbb{C}$ then we take $h : \mathbb{C} \to D$ as $h(z) = z/(1 + |z|)$ and repeat the process above applying Theorem II.2.30.

(j) $\Rightarrow$ (g). Suppose $f : G \to \mathbb{C}$ is analytic and never vanishes on G, and let $u = \text{Re}(f)$ and $v = \text{Im}(f)$. If $U : G \to R$ is defined by $U(x, y) = \log |f(x + iy)| = \log(u(x, y)^2 + v(x, y)^2)^{1/2}$ then "a computation" shows that U is harmonic. By hypothesis (j) there is a harmonic function V on G such that $g = U + iV$ is analytic on G. Let $h(z) = \exp(g(z))$. Then h is analytic on F, h never vanishes on G, and $|f(z)/h(z)| = 1$ for all $z \in G$ since

$$|h| = |\exp g| = |\exp(U + iV)| = \exp U = \exp(\log |f|) = |f|.$$

Theorem VIII.2.2 (continued 11)

Proof (continued). So f/h is analytic on G. By Exercise VI.1.6 there is a constant c of modulus 1 such that

$$f(z) = ch(z) = z\exp(h(z)) = \exp(g(z) + c_1) \text{ where } c = \exp(c_1).$$

So $g(z) + c_1$ is a branch of $\log f(z)$ and (g) holds. □