

## Complex Analysis

## Chapter XI. Entire Functions

## XI.1. Jensen's Formula—Proofs of Theorems



## Lemma XI.1.A

**Lemma XI.1.A.** If  $f$  is analytic in an open set containing  $\bar{B}(0; r)$  and  $f$  doesn't vanish in  $B(0; r)$  then (1.1) holds.

**Proof.** As argued above, the Mean Value Theorem (Theorem X.1.4) gives the result if  $f$  doesn't vanish in  $\bar{B}(0; r)$ . So we only need consider zeros of  $f$  on  $|z| = r$ . We show the result for  $f$  having one zero  $a = re^{i\alpha}$  on  $|z| = r$ . Then by induction, the result holds for  $f$  having a finite number of zeros on  $|z| = r$  (if  $f$  has an infinite number of zeros on  $|z| = r$  then by Theorem IV.3.7  $f \equiv 0$ , but then  $f$  vanishes in  $B(0; r)$ ).

Define  $g(z) = f(z)/(z - a)$  where  $z = re^{i\alpha}$  is the one zero of  $f$  on  $|z| = r$ . Then  $g$  (reduced) has no zeros in  $\bar{B}(0; r)$  and so (1.1) applies to give

$$\begin{aligned} \log |g(0)| &= \frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})| \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left( \log |f(re^{i\theta})| - \log |re^{i\theta} - re^{i\alpha}| \right) \, d\theta. \end{aligned}$$

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Lemma XI.1.A

## Lemma XI.1.A (continued 1)

**Proof (continued).** Now  $\log |g(0)| = \log |f(0)| - \log r$  so

$$\log |f(0)| = \log r = \frac{1}{2\pi} \int_0^{2\pi} (\log |f(re^{i\theta})| - \log |re^{i\theta} - re^{i\alpha}|) \, d\theta. \quad (*)$$

By Exercise V.2.2(h),

$$\int_0^{2\pi} \log(\sin^2(\theta)) \, d\theta = -4\pi \log 2. \quad (**)$$

So

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log |re^{i\theta} - re^{i\alpha}| \, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} (\log r + \log |e^{i\theta} - e^{i\alpha}|) \, d\theta \\ &= \log r + \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i(\theta-\alpha)} - 1| \, d\theta \end{aligned}$$

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Lemma XI.1.A

## Lemma XI.1.A (continued 2)

**Proof (continued).**

$$\begin{aligned} &= \log r \frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| \, d\theta \\ &= \log r + \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} \, d\theta \\ &= \log r + \frac{1}{4\pi} \int_0^{2\pi} \log(2 - 2 \cos \theta) \, d\theta \\ &= \log r + \frac{1}{4\pi} \int_0^{2\pi} (\log 2 + \log(1 - \cos \theta)) \, d\theta \\ &= \log r + \frac{1}{2} \log 2 + \frac{1}{4\pi} \int_0^{2\pi} \log \left( 2 \sin^2 \frac{\theta}{2} \right) \, d\theta \\ &\quad \text{since } \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \end{aligned}$$

## Lemma XI.1.A (continued 3)

**Proof (continued).**

$$\begin{aligned}
 &= \log r + \log 2 \frac{1}{2\pi} \int_0^{2\pi} 4 \log(2 \sin^2 \theta) d\theta \\
 &= \log r + \log 2 + \frac{1}{4\pi} \int_0^{2\pi} \log(2 \sin^2 \theta) d\theta \\
 &= \log r + \log 2 + \frac{1}{4\pi} (-4\pi \log 2) \text{ by (**)} \\
 &= \log r.
 \end{aligned}$$

So by (\*),

$$\log |f(0)| - \log r = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \log r,$$

and (1.1) therefore holds for  $f$  with exactly one zero on  $|z| = r$ . As described above, the claim now holds for  $f$  having a finite number of zeros on  $|z| = r$ .  $\square$

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Theorem XI.1.2. Jensen's Formula

## Theorem XI.1.2 (continued 1)

**Proof (continued).** So

$$F(z) = f(z) \left/ \prod_{k=1}^n \frac{r(z - a_k)}{r^2 - \bar{a}_k z} \right. = f(z) \prod_{k=1}^n \frac{r^2 - \bar{a}_k z}{r(z - a_k)}$$

(when reduced) is analytic on  $\overline{B}(0; r)$ , has no zeros in  $B(0; r)$ , and for  $|z| = r$  we have

$$|F(z)| = \left| f(z) \left/ \prod_{k=1}^n \frac{r(z - a_k)}{r^2 - \bar{a}_k z} \right. \right| = |f(z)|.$$

By by Lemma XI.1.A,

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

Now  $F(0) = f(0) \prod_{k=1}^n (-r/a_k)$ , so

$$\log |F(0)| = \log |f(0)| + \sum_{k=1}^n \log(r/|a_k|) \dots$$

## Theorem XI.1.2

**Theorem XI.1.2. Jensen's Formula.**

Let  $f$  be an analytic function on a region containing  $\overline{B}(0; r)$  and suppose that  $a_1, a_2, \dots, a_n$  are the zeros of  $f$  in  $B(0; r)$  repeated according to multiplicity. If  $f(0) \neq 0$  then

$$\log |f(0)| = - \sum_{k=1}^n \log \left( \frac{r}{|a_k|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

**Proof.** If  $|b| < 1$  then the Möbius transformation  $(z - b)/(1 - \bar{b}z)$  maps the disk  $B(0; 1)$  onto itself and maps the boundary to itself (this follows from the solution to Exercise III.3.10). Replacing  $b$  with  $a_k/r \in B(0; 1)$  and  $z$  with  $z/r$  we have that

$$\frac{z/r - a_k/r}{1 - (\bar{a}_k/r)(z/r)} = \frac{r(z - a_k)}{r^2 - \bar{a}_k z}$$

maps  $B(0; r)$  to  $B(0; r)$  and maps the boundary  $|z| = r$  to  $|z| = 1$ .

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Theorem XI.1.2. Jensen's Formula

## Theorem XI.1.2 (continued 2)

**Theorem XI.1.2. Jensen's Formula.**

Let  $f$  be an analytic function on a region containing  $\overline{B}(0; r)$  and suppose that  $a_1, a_2, \dots, a_n$  are the zeros of  $f$  in  $B(0; r)$  repeated according to multiplicity. If  $f(0) \neq 0$  then

$$\log |f(0)| = - \sum_{k=1}^n \log \left( \frac{r}{|a_k|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

**Proof (continued).** ... and hence

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta =$$

$$\log |f(0)| + \sum_{k=1}^n \log \left( \frac{r}{|a_k|} \right).$$

Jensen's Formula now follows.  $\square$

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## Theorem XI.1.B

**Theorem XI.1.B. Titchmarsh's Number of Zeros Theorem.**

Let  $f$  be analytic in  $|z| < R$ . Let  $|f(z)| \leq M$  in the disk  $|z| \leq R$  and suppose  $f(0) \neq 0$ . Then for  $0 < \delta < 1$  the number of zeros of  $f(z)$  in the disk  $|z| \leq \delta R$  is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

**Proof.** Let  $f$  have  $n$  zeros in the disk  $|z| \leq \delta R$ , say  $a_1, a_2, \dots, a_n$ . Then for  $1 \leq k \leq n$  we have  $|a_k| \leq \delta R$ , or  $\frac{R}{|a_k|} \geq \frac{1}{\delta}$ . So

$$\sum_{k=1}^n \log \frac{R}{|a_k|} = \log \frac{R}{|a_1|} + \log \frac{R}{|a_2|} + \dots + \log \frac{R}{|a_n|} \geq n \log \frac{1}{\delta}. \quad (*)$$

## Theorem XI.1.B (continued)

**Proof (continued).** By Jensen's Formula, we have

$$\begin{aligned} \sum_{k=1}^n \log \frac{R}{|a_k|} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log M d\theta - \log |f(0)| \\ &= \log M - \log |f(0)| \\ &= \log \frac{M}{|f(0)|}. \quad (***) \end{aligned}$$

Combining (\*) and (\*\*\*) gives  $n \log \frac{1}{\delta} \leq \sum_{k=1}^n \log \frac{R}{|a_k|} \leq \log \frac{M}{|f(0)|}$ , or  $n \leq \frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}$ . Since  $n$  is the number of zeros of  $f$  in  $|z| \leq \delta R$ , the result follows.  $\square$