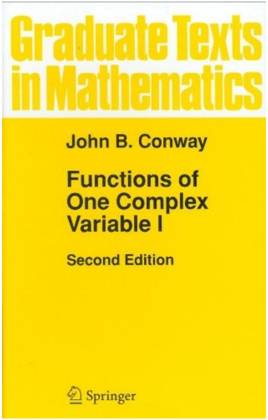
Complex Analysis

Chapter XI. Entire Functions

XI.1. Jensen's Formula—Proofs of Theorems



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Define $g(z) = f(z)/(z-a)$ where $z = re^{i\alpha}$ is the one zero of f on $|z|=r$. Then g (reduced) has no zeros in $\overline{B}(0;r)$ and so (1.1) applies to give

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\log|f(re^{i\theta})| - \log|re^{i\theta} - re^{i\alpha}|\right)\,d\theta.$$

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Proof. As argued above, the Mean Value Theorem (Theorem X.1.4) gives the result if f doesn't vanish in $\overline{B}(0; r)$. So we only need consider zeros of f on $|z| = r$. We show the result for f having one zero $a = re^{i\alpha}$ on $|z| = r$. Then by induction, the result holds for f having a finite number of zeros on $|z| = r$ (if f has an infinite number of zeros on $|z| = r$ then by Theorem IV.3.7 $f \equiv 0$, but then f vanishes in $B(0; r)$).

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Lemma XI.1.A (continued 1)

Proof (continued). Now $\log|g(0)| = \log|f(0| - \log r$ so

$$\log|f(0)| = \log r = \frac{1}{2\pi}\int_0^{2\pi} (\log|f(re^{i\theta})| - \log|re^{i\theta} - re^{i\alpha}|)\, d\theta. \qquad (*)$$

By Exercise V.2.2(h),

$$\int_0^{2\pi} \log(\sin^2(\theta))\, d\theta = -4\pi \log 2. \qquad (**)$$

So

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log|re^{i\theta} - re^{i\alpha}|\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} (\log r + \log|e^{i\theta} - e^{i\alpha}|)\, d\theta \\ &= \log r + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i(\theta-\alpha} - 1|\, d\theta \end{aligned}$$

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Lemma XI.1.A (continued 2)

Proof (continued).

$$
\begin{aligned}
&= \log r \tfrac{1}{2\pi} \int_0^{2\pi} \log \left|1 - e^{i\theta}\right| d\theta \\
&= \log r + \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta \\
&= \log r + \frac{1}{4\pi} \int_0^{2\pi} \log(2 - 2\cos\theta)\, d\theta \\
&= \log r + \frac{1}{4\pi} \int_0^{2\pi} (\log 2 + \log(1-\cos\theta))\, d\theta \\
&= \log r + \frac{1}{2}\log 2 + \frac{1}{4\pi} \int_0^{2\pi} \log\left(2\sin^2\frac{\theta}{2}\right) d\theta
\end{aligned}
$$

$$\text{since } \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Lemma XI.1.A (continued 3)

Proof (continued).

$$\begin{aligned} &= \log r + \log 2 \tfrac{1}{2\pi} \int_0^{\pi/2} 4 \log(2\sin^2\theta)\, d\theta \\ &= \log r + \log 2 + \frac{1}{4\pi} \int_0^{2\pi} \log(2\sin^2\theta)\, d\theta \\ &= \log r + \log 2 + \frac{1}{4\pi}(-4\pi \log 2) \text{ by } (**) \\ &= \log r. \end{aligned}$$

So by $(*)$,

$$\log|f(0)| - \log r = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - \log r,$$

and (1.1) therefore holds for f with exactly one zero on $|z| = r$. As described above, the claim now holds for f having a finite number of zeros on $|z| = r$. □

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Theorem XI.1.2

Theorem XI.1.2. Jensen's Formula.
Let f be an analytic function on a region containing $\overline{B}(0;r)$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity. If $f(0) \neq 0$ then

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Proof. If $|b| < 1$ then the Möbius transformation $(z-b)/(1-\overline{b}z)$ maps the disk $B(0;1)$ onto itself and maps the boundary to itself (this follows from the solution to Exercise III.3.10).

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$$\frac{z/r - a_k/r}{1 - (\overline{a}_k/r)(z/r)} = \frac{r(z - a_k)}{r^2 - \overline{a}_k z}$$

maps $B(0;r)$ to $B(0;r)$ and maps the boundary $|z| = r$ to $|z| = 1$.

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Proof (continued). So

$$F(z) = f(z) \Bigg/ \prod_{k=1}^{n} \frac{r(z-a_k)}{r^2 - \overline{a}_k z} = f(z) \prod_{k=1}^{n} \frac{r^2 - \overline{a}_k z}{r(z-a_k)}$$

(when reduced) is analytic on $\overline{B}(0;r)$, has no zeros in $B(0;r)$, and for $|z| = r$ we have

$$|F(z)| = \left| f(z) \Bigg/ \prod_{k=1}^{n} \frac{r(z-a_k)}{r^2 - \overline{a}_k z} \right| = |f(z)|.$$

By by Lemma XI.1.A,

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Now $F(0) = f(0)\prod_{k=1}^{n}(-r/a_k)$, so

$$\log|F(0)| = \log|f(0)| + \sum^{n} \log(r/|a_k|) \ldots$$

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$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

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$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta =$$

$$\log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right).$$

Jensen's Formula now follows. □

Theorem XI.1.B

Theorem XI.1.B. Titchmarsh's Number of Zeros Theorem.
Let f be analytic in $|z| < R$. Let $|f(z)| \leq M$ in the disk $|z| \leq R$ and suppose $f(0) \neq 0$. Then for $0 < \delta < 1$ the number of zeros of $f(z)$ in the disk $|z| \leq \delta R$ is less than

$$\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

Proof. Let f have n zeros in the disk $|z| \leq \delta R$, say $a_1, a_2, \ldots, a_n$. Then for $1 \leq k \leq n$ we have $|a_k| \leq \delta R$, or $\dfrac{R}{|a_k|} \geq \dfrac{1}{\delta}$.

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$$\sum_{k=1}^{n} \log \frac{R}{|a_k|} = \log \frac{R}{|a_1|} + \log \frac{R}{|a_2|} + \cdots + \log \frac{R}{|a_n|} \geq n \log \frac{1}{\delta}. \qquad (*)$$

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Theorem XI.1.B (continued)

Proof (continued). By Jensen's Formula, we have

$$\begin{aligned}\sum_{k=1}^{n} \log \frac{R}{|a_k|} &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log M\,d\theta - \log|f(0)| \\ &= \log M - \log|f(0)| \\ &= \log\frac{M}{|f(0)|}. \qquad (**)\end{aligned}$$

Combining $(*)$ and $(**)$ gives $n\log\frac{1}{\delta} \le \sum_{k=1}^{n} \log\frac{R}{|a_k|} \le \log\frac{M}{|f(0)|}$, or $n \le \frac{1}{\log 1/\delta}\log\frac{M}{|f(0)|}$. Since n is the number of zeros of f in $|z| \le \delta R$, the result follows. □

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